Math Circles: Number Theory I

Centre for Education in
Mathematics and Computing

February 23, 2011

1 Approximations of π

The following approximation of π is attributed to Archimedes:

$$\pi \approx \frac{22}{7}.$$

Question 1.1. Given that Archimedes was a rational man, how good is this approximation?

Solution. We can certainly suppose that π is closer to $\frac{22}{7}$ than to $\frac{21}{7}$ or $\frac{23}{7}$, since otherwise Archimedes would have given a different approximation. Thus $\frac{22}{7}$ is certainly within $\frac{1}{7}$ of π:

$$\left|\frac{22}{7} - \pi\right| < \frac{1}{7}.$$

Moreover, one can even conclude that π is within $\frac{1}{14}$ of $\frac{22}{7}$. (*Why?*) □

In fact, Archimedes gave considerably more than just an approximation. He gave the bounds:

$$\frac{223}{71} < \pi < \frac{22}{7}.$$

Question 1.2. How good is this approximation?

Solution. We compute:

$$\frac{22}{7} - \frac{223}{71} = \frac{22 \cdot 71 - 7 \cdot 223}{7 \cdot 71} = \frac{1}{7 \cdot 71} = \frac{1}{497}.$$

Thus $\frac{22}{7}$ approximates π to within $\frac{1}{497}$, which is *considerably better than we thought!* □

Suppose you are told that π is closer to $\frac{223}{71}$ than to $\frac{22}{7}$. (It is.)

Question 1.3. Which approximation is better: $\frac{223}{71}$ or $\frac{22}{7}$?

One possible answer. $\frac{223}{71}$ is better, because it's closer. □

Another possible answer. $\frac{22}{7}$ is better, because it's an "unexpectedly good" approximation. One would only expect it to be within $\frac{1}{14}$ of π, but we have seen that it is within $\frac{1}{497}$ of π. In fact:

$$\left|\frac{22}{7} - \pi\right| < \frac{1}{790},$$

which shows that the approximation is even better than what Archimedes implied. □

For this class, we will take the approach that simpler approximations (with smaller denominators) are better. There is some rationale behind this preference: simpler fractions are easier to remember.

Definition 1.4. We say that a rational number $\frac{p}{q}$ is an *unexpectedly good* approximation of a real number α if

$$\left|\alpha - \frac{p}{q}\right|^{-1}$$

is much larger than q.

Question 1.5. How did Archimedes come up with such an unexpectedly good approximation?

Solution. Well, the real answer is that Archimedes used inscribing and circumscribing polygons, but if we were doing this task with modern calculators, we might start with the obvious approximation $\pi \approx 3 = \frac{3}{1}$, and compute as in Definition 1.4:

$$|\pi - 3|^{-1} = \frac{1}{0.14159265358979323 85\dots} = 7.06251330593104577\dots$$

In this way, we find:

$$\frac{1}{\pi - 3} \approx 7,$$

or, solving for π,

$$\begin{aligned} 1 &\approx 7(\pi - 3) \\ \frac{1}{7} &\approx \pi - 3 \\ 3 + \frac{1}{7} &\approx \pi \\ \frac{22}{7} &\approx \pi \end{aligned}$$

□

Question 1.6. How would you go about finding another unexpectedly good approximation of π?

Solution. From above, we have

$$\pi = 3 + \frac{1}{7.06251330593104577\dots}$$

Let's apply the same technique to $7.06251330593104577\dots$. We obtain:

$$7.06251330593104577\dots = 7 + \frac{1}{15.996594406685720\dots} \approx 7 + \frac{1}{16}$$

and thus

$$\pi = 3 + \frac{1}{7.06251330593104577\dots} \approx 3 + \cfrac{1}{7 + \cfrac{1}{16}} = \frac{355}{113}.$$

And indeed, we recognize $\frac{355}{113}$ as another well-known *unexpectedly good* approximation of π:

$$\left|\frac{355}{113} - \pi\right| < \frac{1}{3748629}.$$

□

2 Continued Fractions

We formalize the observations of the preceding section into the following definition.

Definition 2.1. A *continued fraction* is a fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}$$

where each a_i is an integer. We denote the above fraction using the notation $[a_0; a_1, a_2, \ldots, a_n]$.

Given any real number α, we can approximate it via rational numbers using the following sequence of continued fractions:

$a_0 = \lfloor \alpha \rfloor$	$\alpha_1 = \dfrac{1}{\alpha - a_0}$	$\alpha = a_0 + \frac{1}{\alpha_1} \approx a_0$
$a_1 = \lfloor \alpha_1 \rfloor$	$\alpha_2 = \dfrac{1}{\alpha_1 - a_1}$	$\alpha = a_0 + \cfrac{1}{a_1 + \frac{1}{\alpha_2}} \approx a_0 + \cfrac{1}{a_1}$
$a_2 = \lfloor \alpha_2 \rfloor$	$\alpha_3 = \dfrac{1}{\alpha_2 - a_2}$	$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{\alpha_3}}} \approx a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}}$
$a_3 = \lfloor \alpha_3 \rfloor$	$\alpha_4 = \dfrac{1}{\alpha_3 - a_3}$	$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \frac{1}{\alpha_4}}}} \approx a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}}}$

Definition 2.2. The continued fractions above are called the *continued fraction approximations* of α.

Theorem 2.3. *Every continued fraction approximation $\frac{p}{q}$ of α satisfies*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Hence every continued fraction approximation of α is an unexpectedly good *approximation in the sense of Definition 1.4.*

Exercise 2.4. Find the continued fraction approximations of:

1. $\alpha = \sqrt{2}$.
2. $\alpha = \sqrt{3}$.
3. $\alpha = \dfrac{1+\sqrt{5}}{2}$.

Answer. For $\alpha = \sqrt{2}$, we have:

i	a_i	α_i
0	1	$\sqrt{2}$
1	2	$\sqrt{2}+1$
2	2	$\sqrt{2}+1$
3	2	$\sqrt{2}+1$
$\vdots$	$\vdots$	$\vdots$

We see that $a_0 = 1$ and $a_i = 2$ for all $i > 0$. Therefore

$$\sqrt{2} = [1; 2, 2, 2, 2, \ldots] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}$$

The first few continued fraction approximations are:

$$\sqrt{2} \approx 1 = \frac{1}{1}$$
$$\sqrt{2} \approx 1 + \frac{1}{2} = \frac{3}{2}$$
$$\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5}$$
$$\sqrt{2} \approx 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{17}{12}$$

Question 2.5. Do you see any pattern in the sequence of numerators

$$1, 3, 7, 17, \ldots$$

and denominators

$$1, 2, 5, 12, \ldots$$

Answer. Each number is twice the previous number, plus the previous previous number. For example $17 = 2 \times 7 + 3$. □

For $\alpha = \sqrt{3}$, we have:

i	a_i	α_i
0	1	$\sqrt{3}$
1	1	$\frac{\sqrt{3}+1}{2}$
2	2	$\sqrt{3}+1$
3	1	$\frac{\sqrt{3}+1}{2}$
4	2	$\sqrt{3}+1$
5	1	$\frac{\sqrt{3}+1}{2}$
6	2	$\sqrt{3}+1$
$\vdots$	$\vdots$	$\vdots$

Here the terms a_i repeat with periodicity 2:

$$\sqrt{3} = [1;1,2,1,2,\ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}$$

The first few continued fraction approximations are:

$$\sqrt{3} \approx 1 = \frac{1}{1}$$

$$\sqrt{3} \approx 1 + \frac{1}{1} = \frac{2}{1}$$

$$\sqrt{3} \approx 1 + \cfrac{1}{1 + \cfrac{1}{2}} = \frac{5}{3}$$

$$\sqrt{3} \approx 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1}}} = \frac{7}{4}$$

$$\sqrt{3} \approx 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}} = \frac{19}{11}$$

What pattern do you see in the numerators

$$1, 2, 5, 7, 19, \ldots$$

and denominators

$$1, 1, 3, 4, 11 \ldots ?$$

For $\alpha = \dfrac{1+\sqrt{5}}{2}$, we have $\alpha = \frac{1}{\alpha - 1}$ and hence our table becomes:

i	a_i	α_i
0	1	$\frac{1+\sqrt{5}}{2}$
1	1	$\frac{1+\sqrt{5}}{2}$
2	1	$\frac{1+\sqrt{5}}{2}$
3	1	$\frac{1+\sqrt{5}}{2}$
4	1	$\frac{1+\sqrt{5}}{2}$
$\vdots$	$\vdots$	$\vdots$

Thus

$$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}$$

The first few continued fraction approximations are:

$$\frac{1+\sqrt{5}}{2} \approx 1 = \frac{1}{1}$$

$$\frac{1+\sqrt{5}}{2} \approx 1 + \frac{1}{1} = \frac{2}{1}$$

$$\frac{1+\sqrt{5}}{2} \approx 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2}$$

$$\frac{1+\sqrt{5}}{2} \approx 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{5}{3}$$

$$\frac{1+\sqrt{5}}{2} \approx 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}} = \frac{8}{5}$$

$$\frac{1+\sqrt{5}}{2} \approx 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}} = \frac{13}{8}$$

The sequences of numerators $1, 2, 3, 5, 8, 13$ and denominators $1, 1, 2, 3, 5, 8$ should be easily recognizable. What are they? □

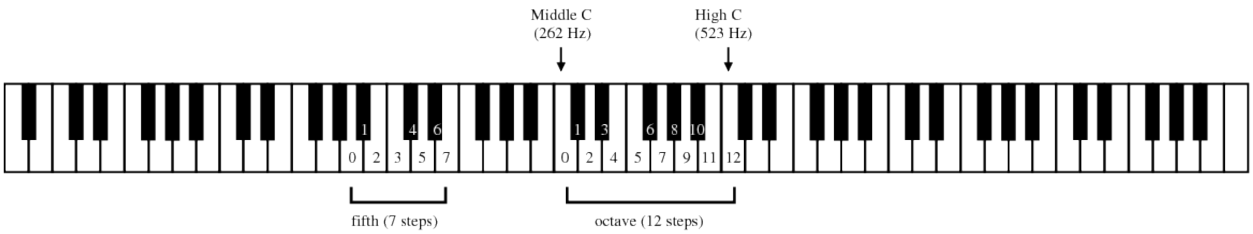

Figure 1: Piano keyboard

3 A musical interlude

For a practical (and surprising) application of rational approximations, we examine the design of the western musical scale.

We consider the sequence of tones produced by a piano keyboard (see Figure 1). Such a sequence of tones is defined to be a *scale*. For our purposes, the necessary musical background consists of the following:

- Each key plays a note of a certain pitch. Keys to the right correspond to notes of higher pitch.
- Each black key is intermediate in pitch from its two adjacent white keys.
- For each of the 88 keys, the ratio in pitch between a given key and the key of the next higher note should be constant.

In addition, there are two additional constraints: the scale must contain *octaves* and *fifths*. An octave is a pair of keys with a 2 : 1 ratio in pitch. A fifth is a pair of keys with a 3 : 2 ratio in pitch.

Traditionally, on a western scale, an octave consists of an interval of 12 keys, and a fifth consists of an interval of 7 keys (see Figure 1). Thus, for example, the key labeled "Middle C" has a pitch of 261.626 Hz, and the key labeled "High C" (one octave above Middle C) has a pitch of $523.251\,\text{Hz} = 2 \times 261.626\,\text{Hz}$.

Unfortunately, the above requirements are contradictory, and cannot be simultaneously satisfied. For example, since an interval of 12 keys forms an octave, the (constant) pitch ratio β between adjacent keys must satisfy $\beta^{12} = 2$. From this, we can calculate

$$\beta = \sqrt[12]{2} = 1.05946309435929526 46\ldots$$

On the other hand, if an interval of 7 keys forms a fifth, then $\beta^7 = 3/2$, or

$$\beta = \sqrt[7]{\frac{3}{2}} = 1.0596340226670483814\ldots$$

These two numbers are almost, but *not exactly*, the same. Thus, a pitch ratio which allows for perfect octaves will preclude perfect fifths, and vice versa. This holds true for *any* scale design:

Theorem 3.1. *Any musical scale with a constant pitch ratio β cannot simultaneously contain perfect octaves and perfect fifths.*

Proof. Suppose that a scale contains:

- constant pitch ratio β,
- n keys in an octave,
- m keys in a fifth.

Then $\beta^n = 2$ and $\beta^m = 3/2$, so

$$\log \beta^n = \log 2$$
$$\log \beta^m = \log \frac{3}{2}$$

or

$$n \log \beta = \log 2$$
$$m \log \beta = \log \frac{3}{2}$$

From this, we conclude

$$\frac{n}{m} = \frac{\log 2}{\log \frac{3}{2}}.$$

But $\alpha = \log 2 / \log \frac{3}{2}$ is an irrational number (why?), so it cannot equal a rational number n/m. □

The proof of Theorem 3.1 indicates that, in order to construct a good musical scale, we need a good *rational approximation* n/m to the irrational number $\alpha = \log 2/\log \frac{3}{2}$. Such a rational approximation would lead to a scale design with n keys in an octave and m keys in a fifth.

Exercise 3.2. Find the first few continued fraction approximations of

$$\alpha = \frac{\log 2}{\log \frac{3}{2}}.$$

Answer. The first few terms of the continued fraction are

$$\alpha = [1; 1, 2, 2, 3, 1, 5, \ldots]$$

and the first few continued fraction approximations are:

$$1, 2, \frac{5}{3}, \frac{12}{7}, \frac{41}{24}, \frac{53}{31}, \ldots$$

□

Question 3.3. Why does the western musical scale have 12 keys in an octave and 7 keys in a fifth, instead of:

1. 5 keys in an octave, and 3 keys in a fifth?
2. 41 keys in an octave, and 24 keys in a fifth?

Partial answer. A five-tone musical scale is used in some cultures, and such a scale is called a *pentatonic* scale.

We don't use scales with 41 keys in an octave, because such a keyboard would be physically impossible for a human to play. (Thus we have a real-world illustration of why rational approximations with small denominators are better.) □

Question 3.4. Given that the design constraints of a musical scale are impossible to satisfy simultaneously, what compromises have been made in the actual western musical scale that musicians use today?

4 Exercises

1. What are the first few terms of the continued fraction of $e = 2.7182818\ldots$?

2. A *quadratic irrational* is an irrational number α of the form

$$\alpha = \frac{r + s\sqrt{d}}{t},$$

where r, s, t, d are integers, $d > 0$. Prove that the terms a_i in the continued fraction

$$\alpha = [a_0; a_1, a_2, a_3, \ldots, a_i, \ldots]$$

are eventually periodic if and only if α is a quadratic irrational.

3. Find the value of each of the following infinite periodic continued fractions. The bar denotes the repeated portion, so for example

$$[1; \overline{1, 2, 3}] = [1; 1, 2, 3, 1, 2, 3, \ldots].$$

   (a) $[1; \overline{1, 2, 3}]$
   (b) $[\overline{1, 2, 3}]$
   (c) $[\overline{3, 2, 1}]$
   (d) $[\overline{1, 2}]$
   (e) $[\overline{2, 1}]$
   (f) $[\overline{2, 3, 1}]$
   (g) $[\overline{1, 3, 2}]$
   (h) $[\overline{1, 2, 2, 2, 1, 2, 24, 2}]$
   (i) $[\overline{2, 24, 2, 1, 2, 2, 2, 1}]$

4. Find the continued fraction representations of each of the following numbers. Note: these are all eventually periodic, by problem 2.

   (a) $\sqrt{7}$.
   (b) $\sqrt{13}$.
   (c) $\sqrt{23}$.
   (d) $\sqrt{103}$.

5 Selected solutions & more exercises

1. $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, \ldots]$. Exercise (hard): Prove that this pattern holds for the entire sequence.

2. *Sketch of one direction of the proof:* If α is a quadratic irrational, then each α_i is of the form

$$\alpha_i = \frac{r_i + s_i\sqrt{d}}{t_i}$$

where $0 \le r_i, s_i, t_i < d$. In particular, there are only finitely many possibilities for α_i, so the α_i (and hence a_i) must eventually repeat.

3. (a) Write

$$[1;\overline{1,2,3}] = 1 + \frac{1}{[\overline{1,2,3}]}$$

and use the result from part (b) to obtain $[1;\overline{1,2,3}] = \dfrac{\sqrt{37}-1}{3}$.

(b) Let $\alpha = [\overline{1,2,3}]$. Then $\alpha = [1,2,3,\alpha]$, or

$$\alpha = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\alpha}}}.$$

Solving for α, we find $7\alpha^2 - 8\alpha - 3 = 0$, or $\alpha = \dfrac{4+\sqrt{37}}{7}$.

(c) Let $\alpha = [\overline{3,2,1}]$. We find that $3\alpha^2 - 8\alpha - 7 = 0$, or $\dfrac{4+\sqrt{37}}{3}$.

(d) $\dfrac{1+\sqrt{3}}{2}$.

(e) $1+\sqrt{3}$.

Exercise: What is the relationship between $[\overline{1,2,3}]$ and $[\overline{3,2,1}]$? Between $[\overline{1,2}]$ and $[\overline{2,1}]$? (Hint: Look at their minimal polynomials.)

4. (a) $\sqrt{7} = [2;\overline{1,1,1,4}]$.

(b) $\sqrt{13} = [3;\overline{1,1,1,1,6}]$.

(c) $\sqrt{23} = [4;\overline{1,3,1,8}]$.

(d) $\sqrt{103} = [10;\overline{6,1,2,1,1,9,1,1,2,1,6,20}]$.

Exercise (hard): Prove that the bar always spans all but the first term.

Exercise (hard): Prove that the last term underneath the bar is always equal to twice the value of a_0.

Exercise (hard): Prove that the numbers underneath the bar, excluding the last term, always form a palindrome. (A *palindrome* is a sequence which reads the same forwards or backwards.) Hint: Use the Exercise from part 3.