Math Circles:
Invariants Part 2

March 30, 2011

For this lecture, we will be working over a few problems from last week as a review before going into new stuff.

First of all, don't forget that an Invariant is something that doesn't change. These problems can take on many different forms - the tricky part is finding the invariant in each case. Let's start off with the following problem:

Example 1: Blackboard Game

I write the numbers 1 through 10 on the blackboard. On each turn, take two numbers, and replace it by their positive difference. Eventually I will only have one number left, will it be even or odd?

Solution:

First of all, there are 5 even numbers and 5 odd numbers on the board. On any turn, I can take 2 even numbers, 2 odd numbers, or 1 even and 1 odd number.

- 2 even numbers: In this case, the difference of the numbers will always be even. So after erasing 2 even numbers, and writing one even number on the board, what is the net change? -1 even number, and unchanged odd numbers.
- 2 odd numbers: Subtracting two odd numbers will yield an even number. The net change is then: +1 even numbers, - 2 odd numbers
- 1 even and 1 odd number: Subtracting an even and an odd number will always get an odd number. The next change is then: -1 even numbers, and 0 odd numbers.

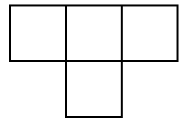
Notice that the number of even numbers on the board can move up or down by 1, but the next of odd numbers can only decrease by 2. What does this mean? If we started with an even number of odd numbers on the board - then we will always have an even number of odd numbers on the board. Similarly, if we start with an odd number of odd numbers on the board, then we will always have an odd number of odd numbers on the board since we can only decrease by two.

Therefore, if started with 5 even and 5 odd numbers, the last number on the board must be odd.

Other places you will see invariants used is in coloring problems. Let's look at a couple from last week:

Example 2: Tetrominos

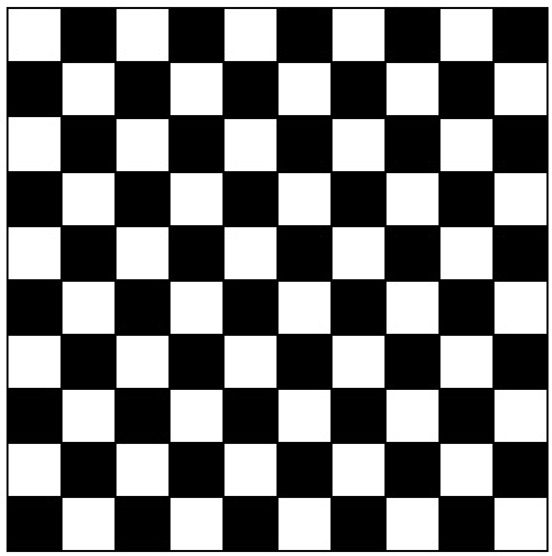
Prove that a 10×10 board cannot be covered by T-shaped tetrominos (shown below).



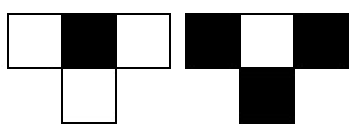
Solution:

A 10×10 board contains 100 squares, and each T-tetromino covers 4 squares. Therefore, we will need 25 T-shaped tetrominos to cover the entire thing.

Let's start by coloring the board in the traditional chessboard pattern - alternating black and white squares, shown below. Notice that there are an equal number of black and white squares, there are 50 of each.



Let's look at all the possible ways you can put a T-shaped tetromino on the board. After a little trial and error, you will realize that there are only two possible cases:

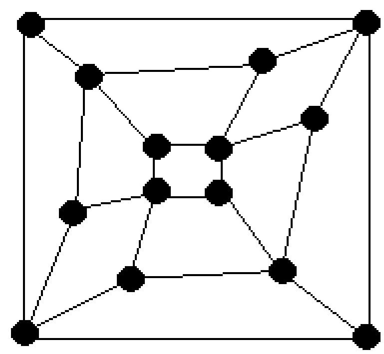


A T-shaped tetronimo can either cover 3 black squares and 1 white square, or 3 white squares and one black. Therefore, in order to cover 50 black squares and 50 white squares, we will need the same number of each type of tetromino to balance the whites and blacks. Since it will take 25 to cover the 10×10 floor, this is impossible!

While chessboards are the most common types of problems to do with coloring, the following is a slight variant.

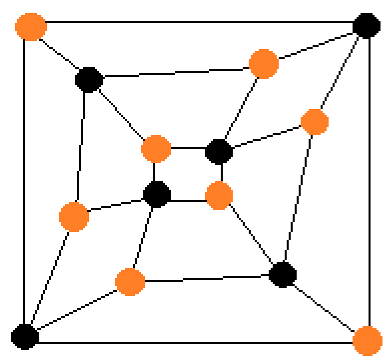
Example 3: Travelling Salesman

The figure shows a map of 14 cities. Can you follow a path that passes through each city exactly once?



Solution:

Start by coloring the map such that neighboring cities will always be different colors.



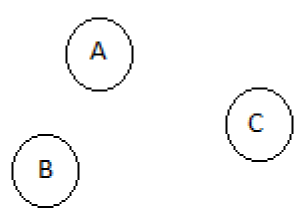
In order to create a path that passes through each city exactly once, notice that your path will have to alternate colored cities each time. So if you start on a black city, then you will move to an orange city, then black, then orange, and vice versa. You will end on an orange city. However, if you look at the graph, there are 8 orange cities, and 6 black ones. This means that while trying to visit each city exactly once, it will be impossible - you will run out of black cities.

Example 4: Hockey!

A hockey player has 3 pucks labelled A, B, C in an arena (as in all math, an infinite area!). He picks a puck at random, and fires it through the other 2. He keeps doing this. Can the pucks return to their original spots after 2011 hits?

Solution:

To start, suppose the pucks begin in the order ABC, clockwise, like below.

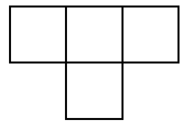


You can either shoot A through B and C, shoot B through A and C, or C through A and B. In all three cases, notice that the orientation of the pucks, clockwise, is always going to be ACB, as opposed to ABC. Moving another puck after this will rearrange the pucks back to the ABC orientation. Therefore, every odd numbered move, i.e. 1, 3, 5, etc, will end in the orientation ACB, and every even numbered move will end in ABC. Therefore, after 2011 moves, the pucks orientation will be ACB, and hence we can never end up where we started.

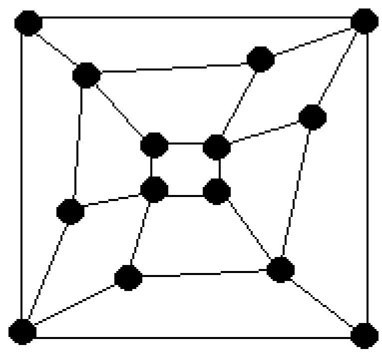
Invariance Part 2: Problems

1. I write the numbers 1 through 10 on the blackboard. On each turn, take two numbers, and replace it by their positive difference. Eventually I will only have one number left, will it be even or odd?

2. Prove that a 10×10 board cannot be covered by T-shaped tetrominos (shown below).



3. The figure shows a map of 14 cities. Can you follow a path that passes through each city exactly once?



4. A hockey player has 3 pucks labelled A, B, C in an arena (as in all math, an infinite area!). He picks a puck at random, and fires it through the other 2. He keeps doing this. Can the pucks return to their original spots after 2011 hits?

Invariance Part 2: Problem Set

1. We have a $m \times n$ rectangular chocolate bar, with lines where you can break the chocolate bar. You can break the chocolate bar along predetermined lines only. How many breaks will it take to break the chocolate bar into mn pieces?
2. There are 13 green, 15 blue, and 17 red chameleons on Camelot Island. Whenever two chameleons of different colors meet, they both swap to the third color (i.e., a green and blue would both become red). Is it possible for all chameleons to become one color?
3. An $m \times n$ table is filled with numbers such that each row and column sum up to 1. Prove that $m = n$.
4. Assume an 8×8 chessboard, in the usual coloring. You can repaint all the squares of a row or column, i.e., all white squares become black, and all white squares become black. Can you get exactly one black square?
5. A rectangular floor is covered by 2×2 and 1×4 tiles. One tile got smashed, but we have one more tile of the other kind available. Can we retile the floor perfectly?
6. Show that an 8×8 chessboard cannot be covered by 15 T-tetrominoes and one square tetromino (a 2×2 tile).
7. There is a positive integer in each square of a rectangular table. In each move, you may double each number in a row, or subtract 1 from each number of a column. Prove that you can reach a table of zeros by using only these moves.
8. There is a town where all its inhabitants are dwarves. All friendships are mutual, if Susan is friends with Dave, then Dave is also friends with Susan. Every month, a dwarf goes outside and looks at all his/her's friends' houses. If their house doesn't match the majority (or one of the ones tied with the majority), then they will change their house color to match. Prove that eventually no more houses will be repainted.
9. Everyone who attends Math Circles has at most three friends here. Prove that Math circles can be divided into two groups, so every one has at most one friend in their group.
10. Prove that at any given time, two diametrically opposite places on the Earth have the same temperature.