UNIVERSITY OF WATERLOO
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Faculty of Mathematics

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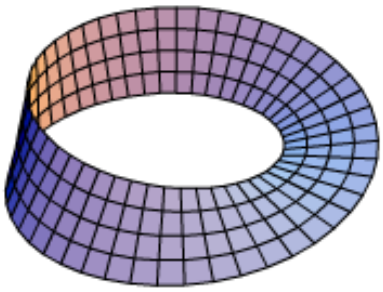
Grade 6 Math Circles
November 16, 2011
Introduction to Topology

Topology is the study of shapes. More specifically, it is the study of what about shapes stays the same when they are stretched or twisted.
If one shape can be stretched or twisted into another shape, then in topology, the two shapes are considered to be the same. For this reason, the size and proportion of shapes have no meaning in topology, what matters is how many holes and twists a shape possesses. A rectangle the size of your house is the same as a playing card. A coffee mug is the same as a donut.

One of the most interesting topological shapes is the *Mobius band* or *Mobius strip*, which is named after the German mathematician and astronomer August Ferdinand Mobius (Mobius did not actually discover the band, the mathematician Johan Benedict Listing was the first to describe it in 1861).

How to make a Mobius band



Take a strip of paper and twist one of the ends 180°, now tape the two ends together.
Be sure to tape all the way across the edge, or else your band will come apart later.

Investigating the Mobius band

How many sides does a Mobius band have? Make a hypothesis, then see if you are right by drawing a line down the middle of your Mobius band with a pencil. Do not lift your pencil until you have reached your original starting point. How many lines did you have to draw to cover both sides of your original paper? What does this tell you about the number of sides on a Mobius band?

We can conduct a similar investigation to determine the number of edges. Run your pencil along one edge of your Mobius band. Do not stop until you return to your starting point. How many times did you have to lift your pencil in order to completely trace the perimeter of your Mobius band? Do you see results similar to the previous investigation? How many edges are there in a Mobius band?

Conclusions

We have discovered that the Mobius band is a shape with one side and one edge. This means that an ant could completely cover the area of our band (both sides of the original paper) without ever crossing an edge! Artist M.C. Escher used this fact in his woodcut titled "Mobius Strip II", which depicts nine ants walking around a Mobius band.

Cutting the Mobius band

Even though topology does not cover what happens to shapes when they are cut, we can still make some very interesting discoveries by cutting into the Mobius band:

Experiment 1
Make a hypothesis as to what happens when we cut a Mobius band in half. Try it and see what you get! Make sure you begin cutting by making a small hole in the middle of your band, and not by cutting into the edge. Was your hypothesis correct?

Experiment 2
Make a hypothesis as to what will happen when you cut your result from the previous experiment in half again. Try it out! How accurate was your hypothesis?

Experiment 3
Make a new Mobius band. In this experiment we are interested in what happens when we cut our Mobius band starting from one-third of the way across (instead of halfway across as in the first experiment). Make a hypothesis as to what will happen and see how accurate you are.

Extensions

Do the properties of the Mobius band stay the same if we construct it with more than one half-twist? Make three more Mobius bands, one with two half-twists, one with three, and one with four. Then complete the following table:

Number of half-twists	Number of sides	Number of edges	What happens when cut in half
2			
3			
4			

Write down any patterns that you see:

Applications of the Mobius band

These crazy topological shapes even have uses in the real world!

Mobius bands have been used as conveyor belts that last longer because the entire surface area of the belt gets the same amount of wear, and as continuous-loop recording tapes to double the playing time without having to flip over the tape.

Mobius bands are common in the manufacture of fabric computer printer and typewriter ribbons, as they allow the ribbon to be twice as wide as the print head while using both half-edges evenly.

In the 1960's Sandia Laboratories used Mobius bands in the design of versatile electronic resistors.

3-Dimensional Mobius bands: The Klein bottle

In 1882, German mathematician Felix Klein imagined sewing two Mobius Loops together to create a single sided bottle with no boundary. This is only possible in 4-dimensonal space. Its inside is its outside. It contains itself. It's also *non-orientable*, which means a symbol on its surface can be slid around on it and reappear backwards at the same place, something not possible with 3-dimensional shapes (try it!).

A true Klein bottle only exists in 4 dimensions, however, we can make a 3-D visualization of a 4-D Klein bottle (much like a picture of a 3-D object is a 2-D visualization of that object).

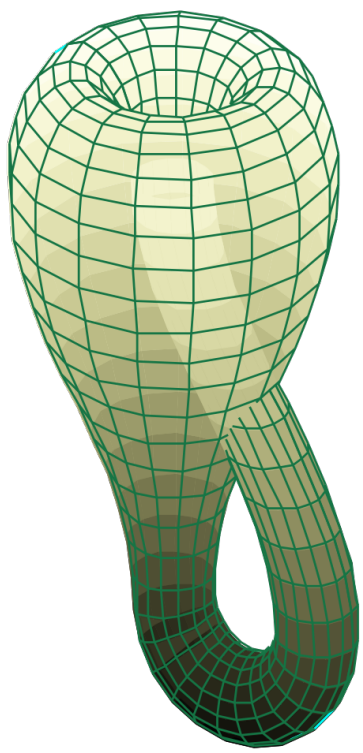


Image: Tttrung, http://commons.wikimedia.org/wiki/File:Klein_bottle.svg, Nov 9 2011

In 4-dimensional space, the bottle would actually pass through itself without the use of a hole.

Exercises

1. **Mobius' Crosses:** To begin, take two strips of paper and tape them together at a right angle so they make a "+". Be sure to securely tape all of the edges. You will need three of these crosses altogether.

   (a) With your first cross, take two opposite arms and tape them together in an untwisted loop. Take the other two arms and tape them together in an untwisted loop. Your cross should now look like a twisted figure-8.
   What do you think will happen when you cut both loops down their middles? Try it to test your hypothesis.

   (b) Using your second cross, again tape opposite ends into loops, but this time make one loop and one Mobius band. What do you think will happen when we cut these loops down their middles? Test your hypothesis.

   (c) With your third cross, tape both pairs of opposite ends into Mobius bands. Make a hypothesis as to what will happen when they are cut in half and test your hypothesis (Note: depending on how you twisted your loops, a couple of different outcomes can happen, don't worry if your cross turns out different than your friend's).

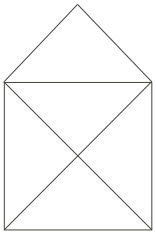
2. **Double Mobius Bands:** Hold two strips of paper together in your hand (one underneath the other), and bring the ends together with a half-twist (like you are making a Mobius band). Get a friend to help you tape the top two strips together and the bottom two strips together. You should now have two Mobius bands nestled snugly side by side.

   (a) Take a pencil and put it between the two loops. Move the pencil around the loop one time until it returns to the place where you began. Is your pencil facing the same or opposite direction than it was before? Can you explain this? What happens when you move your pencil around the loop a second time?

   (b) Make a hypothesis as to what will happen when we pull the two bands apart. Test your hypothesis.

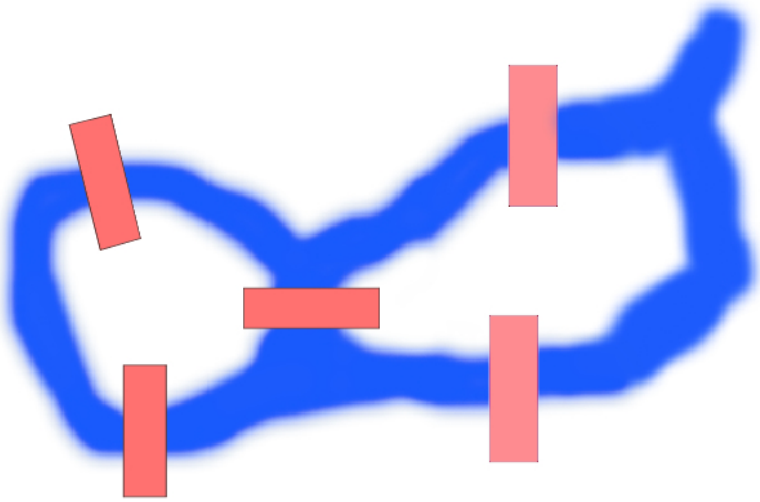
Puzzles

1. Draw the "house" figure on the right, without lifting your pencil and without crossing over or retracing any lines.



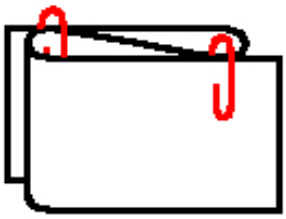
2. Take a small piece of paper (at most one quarter of a full sheet) and in the middle trace around the rim of a penny. Then carefully and accurately cut a circular hole in your paper along that outline. Your challenge now is to pass a quarter through this hole without tearing the paper.

3. Rachel's town has two islands connected through five bridges. Rachel likes to take a walk around the islands by crossing every bridge once, and only once. Trace two of the possible routes Rachel can follow on her walk. Where must Rachel start her walk?



4. For this puzzle, use half of a piece of paper. Your challenge is to cut a hole in this paper so large that a person can fit through it.
Hint: Cutting a hole does not necessarily mean we have to remove parts of the paper. How big of a hole can we get by cutting an "H" shape into the paper? Now make a vertical cut in the centre of the top edge of your paper, stopping just before the horizontal line of your "H". Make a similar vertical cut in the bottom edge of your paper. Is the hole bigger or smaller than before? Can we extend this?

5. Using a strip of paper and two paperclips, fold the strip over and then back, so you get a "Z" shape, like the picture to the right. Now slide the paperclips on from one side, so that the paper is held in this shape.



   (a) Make a hypothesis as to how far apart the paperclips will land when you pull hard on both ends of the paper, making them fly into the air. Try it out, are you surprised by your results?

   (b) Try to achieve the same result using three or more paperclips, and more than two folds.

6. Ask your teacher for a copy of the "Hanging Boots Puzzle". Once you have solved it, return it to your teacher.

7. Try to recreate the figure below using a single sheet of paper and no tape.