University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles
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Matrices I

When we are working on math problems, we are used to seeing one expression at a time, for example:

$$6 + 5 \qquad \text{or} \qquad 2 \times 3 + 1$$

But sometimes it is useful for us to do the same set of operations on different numbers at the same time, for example:
$8 + 12 - 11$
$37 + 4 - 19$
$21 + 43 - 58$
$17 + 23 - 33$
A compact method of writing all of these expressions at once is using a ***matrix***, which is a rectangular arrangement of mathematical expressions (usually numbers). Matrices can be any rectangular size, for example:

2×2: $\begin{bmatrix} 4 & 1 \\ 6 & 9 \end{bmatrix}$

2×3: $\begin{bmatrix} 17 & 51 & 66 \\ 45 & 10 & 22 \end{bmatrix}$

3×1: $\begin{bmatrix} 6 \\ 13 \\ 47 \end{bmatrix}$

Notice that when we write the dimensions of a matrix, we write
number of rows $\times$ *number of columns*

The expressions inside matrices are called "entries" and are typically labelled by their row, and then their column.
For example, in the 2×2 matrix above, the number 1 is the 1,2 entry as it is in the first row and the second column. Similarly 6 is the 2,1 entry.

When matrices themselves are labelled it is usually using capital letters, and the entries can then be labelled with lower case letters using subscripts.
For instance, if we were to label our above 2×2 matrix A, we could then label the numbers 1 as $a_{1,2}$ and 6 as $a_{2,1}$

Addition and Subtraction of Matrices

If we want to add or subtract multiple matrices, they must all have the same dimensions. We then simply perform the operations separately, entry by entry. Your answer will also be a matrix of the same size as those in the question.

Examples

The following examples use the matrices $A=\begin{bmatrix} 9 & 13 \\ 36 & 23 \end{bmatrix}$, $B=\begin{bmatrix} 15 & 42 \\ 29 & 55 \end{bmatrix}$, and $C=\begin{bmatrix} 4 & 26 \\ 8 & 11 \end{bmatrix}$

Compute:

1. $A+B$

2. $B-C$

3. $(B+A)+C$

4. $A+(B+C)$

We can now see that 2×2 matrices are just another way to write four similar expressions at once.

Example 5

Use 2×2 matrices to represent the four expressions from the beginning of the lesson. Simplify your expression (write as a single matrix). What other sizes of matrices could we have used to represent these expressions?

Scalar Multiplication
We can multiply a matrix by any number that is not zero (called a *scalar*) by simply multiplying every entry in the matrix by the scalar. When combining scalar multiplication with addition or subtraction remember to follow the standard order of operations.

Example 6
Compute the following:

(a) $3\times\begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$

(b) $2\times\begin{bmatrix} 6 & 10 \\ 13 & 25 \end{bmatrix}+\begin{bmatrix} 11 & 2 \\ 20 & 17 \end{bmatrix}$

(c) $2\times\left(\begin{bmatrix} 1 & 14 \\ 9 & 7 \end{bmatrix}+\begin{bmatrix} 4 & 22 \\ 10 & 5 \end{bmatrix}\right)$

(d) $10\times\left(2\times\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}+3\times\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$

Matrix Multiplication

Multiplying a matrix by another matrix is not as intuitive as other matrix operations, and is only allowed under certain conditions.

If we want to multiply two matrices A and B in the order $A \times B$ (we will soon find out that order matters when multiplying matrices), we must first make sure that **the number of columns in A is equal to the number of rows in B.**

An easy way to check this is to line up the dimensions of our matrices side by side. If the two middle numbers are the same, then the matrices can be multiplied.

ex. If $A=\begin{bmatrix} 15 & 42 \\ 7 & 34 \\ 29 & 55 \end{bmatrix}$ *and* $B=\begin{bmatrix} 12 & 76 & 99 & 11 \\ 2 & 68 & 43 & 28 \end{bmatrix}$,

then A is 3×2 and B is 2×4,

and $3 \times \mathbf{2}$ by $\mathbf{2} \times 4$ passes our "matching middles" test, so A and B can be multiplied!

Notice how our test tells us that the number of columns in A and the number of rows in B are the same.

Also notice that if we were to switch the order of our matrices and multiply $B \times A$, our test would not pass. ***Order matters when multiplying matrices!***

Example 7.1

Using the matrices $A=\begin{bmatrix} 45 & 32 \\ 297 & 85 \end{bmatrix}$, $B=\begin{bmatrix} 15 & 24 & 43 \\ 18 & 65 & 39 \\ 90 & 88 & 5 \end{bmatrix}$, $C=\begin{bmatrix} 5 & 2 \\ 1 & 22 \\ 35 & 7 \\ 29 & 55 \end{bmatrix}$, and $D=\begin{bmatrix} 47 \\ 81 \\ 95 \end{bmatrix}$,

determine which of the following are allowed to be multiplied:

(a) $A \times B$

(b) $B \times A$

(c) $B \times D$

(d) $D \times B$

(e) $A \times C$

(f) $C \times A$

So now we know when we can multiply matrices, but how do we actually do it? Is our result still a matrix?
The answer to this question is yes. If we can multiply two matrices A and B, our result will be a matrix (which we'll call AB) that has the same number of rows as A and the same number of columns as B.
Looking back to our "matching middles" test, if we remove the middle two numbers (which are the same as we can multiply these matrices), the two numbers we are left with are the dimensions of our new matrix. So, in the above example, multiplying would result in a 3×4 matrix.

Example 7.2
From Example 7.1, if the matrices were allowed to be multiplied, find the dimensions of the resulting matrix.

How To Multiply Matrices

When multiplying two matrices A and B, to find the m,n entry of their product AB, we look at the mth row of A and the nth column of B. Because of our previous test, these will have the same number of entries.

Multiply the first entry of A's row by the first entry in B's column, set this answer aside.

Multiply the second entry of A's row by the second entry of B's column, set this aside as well.

Repeat with all other remaining entries in the row/column. Once you have all of your products, add them together to get the m,n entry of AB.

Repeat these steps for all of the entries in AB.

Example 8
Compute $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$.

Exercises

1. Simplify

   (a) $\begin{bmatrix} 44 & 31 & 16 & 29 \end{bmatrix} + \begin{bmatrix} 66 & 19 & 76 & 27 \end{bmatrix}$

   (b) $\begin{bmatrix} 11 & 24 & 81 \\ 54 & 39 & 75 \end{bmatrix}$ - $\begin{bmatrix} 7 & 16 & 65 \\ 41 & 20 & 66 \end{bmatrix}$

   (c) $4\times \begin{bmatrix} 6 & 14 \\ 21 & 9 \end{bmatrix}$

   (d) $2 \times \left(3 \times \begin{bmatrix} 8 & 12 \\ 7 & 10 \\ 5 & 13 \end{bmatrix} + \begin{bmatrix} 11 & 15 \\ 30 & 4x \\ 8 & 7x \end{bmatrix} \right)$

2. Compute the following matrix products, if they exist:

   (a) $\begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 4 & 10 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 5 \\ 4 & 10 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix}$

   (c) $\begin{bmatrix} 11 & 14 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 12 & 2 \\ 13 & 5 \end{bmatrix}$

   (d) $\begin{bmatrix} 5 & 6 \\ 12 & 2 \\ 13 & 5 \end{bmatrix} \begin{bmatrix} 11 & 14 \\ 6 & 7 \end{bmatrix}$

   (e) $\frac{1}{2}\times \begin{bmatrix} 4 & 12 \\ 8 & 11 \end{bmatrix} \begin{bmatrix} 6x & 3 \\ 10x & 5 \end{bmatrix}$
   Should we do the scalar multiplication first or the matrix multiplication first? Or does it matter?

3. Mary and Cecilia are going shopping for fruit to feed their hungry families. Mary needs 3 bags of oranges and 4 bags of apples. Cecilia needs 5 bags of oranges and 2 bags of apples.

   (a) Arrange this information in a 2×2 matrix using the rows to represent the ladies, and the columns to represent the fruit. Call this matrix A

   (b) The grocery store sells bags of oranges for 3$, and each bag weighs 1 kg. They also sell 5$ bags of apples that weigh 2 kg. Arrange this information in a 2×2 matrix using the rows to represent the fruit and the columns to represent the price and weight. Call this matrix B

   (c) Multiply $A \times B$. What does each entry represent?

4. You and a friend start making sandwiches at sell to your school. You decide on two types of sandwiches to sell:

   1. bread, ham, and cheese
   2. bread, cheese, and tomato

   (a) Each day, you plan to sell 4 type 1 sandwiches and 5 type 2 sandwiches. Your friend plans to sell 3 sandwiches of each type. Arrange this information in a 2×2 matrix using rows to represent you and your friend, and columns to represent the types of sandwiches.

   (b) Each sandwich contains some amount of bread, ham, cheese, or tomato. Sandwich 1 is made from 2 slices of bread, 3 slices of ham, and 2 slices of cheese. Sandwich 2 is made from 2 slices of bread, 4 slices of cheese and 2 slices of tomato. Arrange this information in a 2×4 matrix using rows to represent the sandwich types and columns to represent the ingredients.

   (c) Use the previous two matrices to determine how much of each ingredient you and your friend will each need to make one day's worth of sandwiches.

   (d) Bread costs \$0.12 per slice, ham costs \$0.30 per slice, cheese costs \$0.25 per slice, and tomato costs \$0.12 per slice. Arrange this information in a 4×1 matrix, and multiply one of your previous answers by this matrix (ie. this matrix should be on the right) to determine the cost of making one sandwich of each type.

   (e) How much money should you and your friend each budget to make one day's worth of sandwiches? Which two matrices did you multiply to find this answer?