University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles
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Fibonacci and the Golden Ratio

You have all encountered patterns somewhere in your life before, maybe you are even wearing a pattern on your shirt right now!
Visual patterns make use of repetition. If you know how the elements repeat you can use this knowledge to extend the pattern. For example,
Find the next two terms of this pattern: ♡♠♣♡♠♣♡

Numbers can be used to make patterns as well. Numeric patterns don't typically use repeating numbers, but every term in the pattern follows a certain rule, which helps us to extend the pattern. Patterns of numbers are often called *sequences*.

Example 1

Find the next two terms of each sequence, and state each sequence's rule.

(a) $5, 10, 15, 20, 25, ...$

(b) $8, 4, 2, 1, 0.5, 0.25, ...$

In part (b), each term in the sequence (except the first) depended on the term before it. These kinds of sequences are called *recursive sequences.*
One of the most famous recursive sequences is the ***Fibonacci Sequence***. The first few terms are given below, see if you can figure out the rule and the next few terms:

(c) $1, 1, 2, 3, 5, 8, 13, ...$

Let's look at a bit of the history behind this sequence...

Leonardo Pisano was an Italian mathematician who was better known by his nickname Fibonacci (Italian for "Son of Bonacci"). He was born around the year 1170 and died sometime around 1250. Besides discovering the Fibonacci Sequence, he is also responsible for introducing the decimal number system to Europeans, which replaced Roman Numerals.

Fibonacci discovered his sequence by investigating the breeding patterns of rabbits. In his book *Liber Abaci*, published in 1202, Fibonacci posed the following problem:

"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"

In other words, once a pair of rabbits is one month old, it gives birth to a new pair of rabbits every month. Fibonacci was very surprised with the pattern he found in his results...

Example 2
Using Fibonacci's rules, determine how many pairs of rabbits there are after 5 months.

This is not the only place in nature where the Fibonacci Sequence is found:

Example 3
Using the following steps, create the *Fibonacci Spiral* on the provided graph paper. Does it look familiar to you?

1. Near the center of your paper, daw two squares with sides of length 1 side-by-side
2. On top of these two squares, draw a square with sides of length 2
3. To the right of these three squares, draw a square with sides of length 3
4. Below your squares, draw a square with sides of length 5
5. To the left of these squares, draw a square with sides of length 8
6. Continue adding squares (their side lengths will all be Fibonacci Numbers) until you run out of room on your paper
7. Start with your pencil in the first square you drew which is now in the center. Draw curving quarter circles through the corners of the squares starting with the center going clockwise and work your way out to the biggest square. These quarter circles should work together to complete a spiral by arching through two diagonal corners of each square.

In addition to being found in rabbit breeding and seashells, Fibonacci can also be found in the florets of sunflowers, the bottom of pine cones, the rind of pineapples, and the petals of many flowers.

The Golden Ratio

Another interesting property of the Fibonacci Sequence is what happens when we divide consecutive terms.

Example 4
Here are the first 14 terms of the Fibonacci Sequence:

1, 1, 2, 3, 5, 8, 13, 11, 34, 55, 89, 144, 233, 377

Let's see what happens when we take two neighbouring terms and divide the bigger number by the smaller number...

$1 \div 1 =$

$2 \div 1 =$

$3 \div 2 =$

$5 \div 3 =$

$8 \div 5 =$

$13 \div 8 =$

$21 \div 13 =$

$34 \div 21 =$

$55 \div 34 =$

$89 \div 55 =$

$144 \div 89 =$

$233 \div 144 =$

$377 \div 233 =$

The number that these quotients are approaching is known as the ***Golden Ratio*** and has a value of $\frac{1+\sqrt{5}}{2} = 1.618033989...$ (The decimal goes on forever).
Mathematicians have named this special number using the greek letter ϕ, which is pronounced "phi". ϕ has some interesting properties...

Example 5
Which of the following equations are true?

(a) $1 \div \phi = \phi - 1$

(b) $\phi \times \phi = \phi + 1$

(c) $\phi \div 2 = 1 \div \phi$

(d) $\phi \times \phi \times \phi = 2 \times \phi + 1$

Approximating ϕ

ϕ is an example of an *irrational* number, which is a number which cannot be expressed as a fraction with only whole numbers. Above we said that the fraction $\frac{1+\sqrt{5}}{2}$ was equal to ϕ, but since $\sqrt{5}$ is not a whole number (it is actually also an irrational number), this makes ϕ irrational as well.
Even if we can't use whole numbers to make a fraction that equals ϕ, we can use them to make a fraction that approximates it:

Example 6

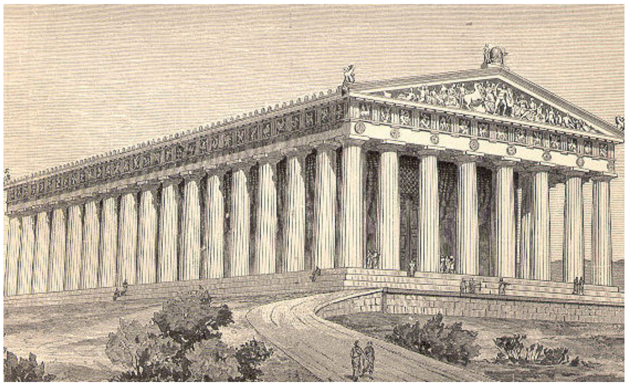
(a) Follow these steps to find an approximation of ϕ using only 1's:

1. Start by adding together $1 + 1$. This is our first approximation for ϕ (it's not very good, is it?).
2. Divide 1 by your previous answer, and then, add 1 to your quotient. This is your new approximation. Is it closer to the actual value of ϕ?
3. Repeat step 2 at least 5 more times, each time using your previous approximation as your denominator. How long until your approximation is correct to three decimal places?

(b) Write out our approximation using only fractions containing 1's and +'s (Hint: It will contain fractions inside of fractions).

ϕ in Architecture

ϕ has long been thought to hold some universal visual appeal. A rectangle where the ratio of $\frac{length}{width} = \phi$ is called a *Golden Rectangle*. These can be found in both ancient and modern architecture, including the Parthenon of Ancient Greece, and the United Nations building.



There are many, many golden rectangles found in the Parthenon.

Image: Stefano Corso
http://commons.wikimedia.org/wiki/File:UN_building.jpg
October 5^{th}, 2011

Every eleven floors of the UN building makes a golden rectangle.

ϕ can also be found in the architecture of the CN tower and the Great Pyramid of Giza in Egypt.

Image: Wladyslaw
http://commons.wikimedia.org/wiki/File:Toronto_-_ON_-_CN_Tower_bei_Nacht2.jpg
October 5^{th}, 2011
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The total height of the tower (553.33m) divided by the height at its observation deck (342m) $\approx \phi$.

Image: Nina Aldin Thune
http://commons.wikimedia.org/wiki/File:Kheops-Pyramid.jpg
October 5^{th}, 2011

If we take half of the base of the pyramid to equal 1, then the slant height of the pyramid $\approx \phi$.

ϕ in the Human Body

You yourself even contain many golden ratios!

Example 7
Find a partner and use the provided measuring tape to take the following pairs of measurements:

(a) From the top of the head to the bottom of the neck, and from the bottom of the neck to the belly button

(b) From the top of the forehead to the bottom of the nose, and from the bottom of the nose to the bottom of the chin

(c) Bend your index finger into a U shape. Now measure from the first knuckle to the second knuckle and from the second knuckle to the third knuckle

For each pair of measurements, divide the bigger number by the smaller number. How close are your proportions to being "golden"?

Exercises

1. **Tribonacci:** Make your own recursive sequence, with each term being the sum of the previous *three* terms. You get to choose what the first three terms in your sequence are (1,1,1 and 1,2,3 are common, or try something wacky like 0,5,23). Write out the first 10-15 terms of your sequence. Can you find any interesting properties? Do the ratios in your sequence approach a special value like ϕ?

2. **Leonardo's Lane:** The City of Waterloo is working on building houses on their newest street, Leonardo's Lane. There are two options for which type of house to build:

    1. A single house which takes up 1 housing lot
    2. Two single houses sharing a common wall (two semi-detached houses) which take up 2 housing lots

    How many different ways can the city build the houses on Leonardo's Lane if they have

    (a) 3 lots?
    (b) 4 lots?
    (c) 5 lots?

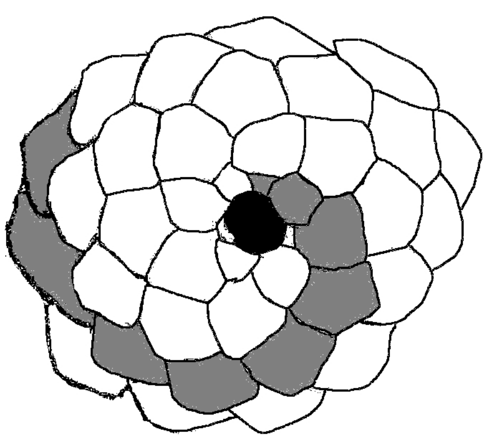
    What kind of pattern is emerging here?

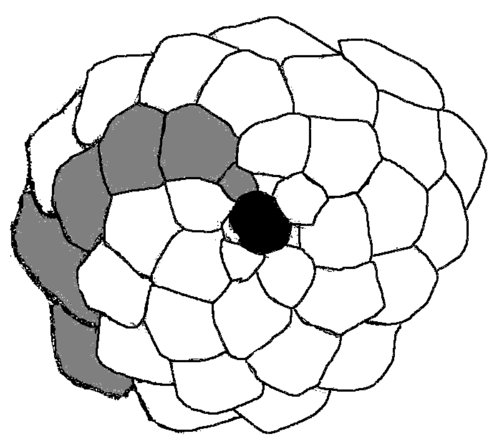
3. **Leonardo's Leaps:** Leo likes to takes the stairs up to his office rather than the elevator. He usually climbs them one step at a time but sometimes he climbs two steps at once. How many different ways can Leo climb the stairs if they contain

    (a) 3 steps?
    (b) 5 steps?
    (c) 7 steps?

    What is the pattern? Can you guess how many ways there are to climb 6 steps?

4. Below are two drawings of the bottom of a pine cone. Going in a clockwise direction, we can find 5 distinct spiral patterns, and going in the counterclockwise direction, there are 8 distinct spiral patterns. Find these patterns in the drawings below. One pattern has been found in each pine cone to help you.





5. Which of the following equations are true?

    (a) $\phi \times \phi \times \phi = \phi \times \phi + \phi$

    (b) $\phi \times \phi \times \phi \times \phi = \phi \times \phi + 2 \times \phi + 1$

    (c) $2 \times \phi = \phi - 2$

6. A triangle can be formed having side lengths 4, 5, and 8. It is impossible, however, to construct a triangle with side lengths 4, 5, and 9. Amadeusz has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

7. Find a partner and use the provided measuring tape to take the following pairs of measurements:

    (a) From the top of the head to the belly button, and from the belly button to the floor

    (b) From the belly button to the bottom of the kneecap, and from the bottom of the kneecap to the floor

    (c) the length of the face, and the width of the face across the eyes

    For each pair of measurements, divide the bigger number by the smaller number. How close are your proportions to being golden?

8. Find a new partner and play Fibonacci Nim! The rules are as follows:

    - Place any number of the provided counters on the table
    - The player who goes first must take at least 1 counter, but can not take all of the counters
    - From then on, a player can take any number of counters that is less than or equal to twice what the previous player took. For example, if your partner has just taken 3 counters, you may take anywhere from 1 to 6 counters.
    - The player who takes the last counter wins!

    Play a few rounds each with a few different partners. Try starting with a different number of counters each time. The winning strategy for this game involves Fibonacci Numbers, can you figure it out?