Math Circles, Solid Geometry

Lesson 1: Vectors, Dot Product and Cross Product

A **point** or **vector** in Euclidean space $\mathbf{R}^3$ is an ordered triple $u = (x, y, z)$ with $x, y, z \in \mathbf{R}$. Given the vector $u = (x, y, z)$ and given $t \in \mathbf{R}$ we define

$$\begin{aligned} t\,u &= t(x, y, z) = (tx, ty, tz) \in \mathbf{R}^3 \\ |u| &= \big|(x, y, z)\big| = \sqrt{x^2 + y^2 + z^2} \in \mathbf{R} \end{aligned}$$

The number $|u|$ is called the **length** of the vector u. Given two vectors $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ we define

$$\begin{aligned} u + v &= (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in \mathbf{R}^3 \\ u - v &= (x_1, y_1, z_1) - (x_2, y_2, z_2) = (x_1 - x_2, y_1 - y_2, z_1 - z_2) \in \mathbf{R}^3 \\ u \bullet v &= (x_1, y_1, z_1) \bullet (x_2, y_2, z_2) = (x_1x_2 + y_1y_2 + z_1z_2 \in \mathbf{R} \\ u \times v &= \big(y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1\big) \in \mathbf{R}^3 \end{aligned}$$

The number $u \bullet v$ is called the **dot product** of u and v, and the vector $u \times v$ is called the **cross product** of u and v.

The dot product has the following properties where $u, v, w \in \mathbf{R}^3$ and $t \in \mathbf{R}$ and where $\theta(u, v)$ is the **angle** between u and v

$$\begin{aligned} & u \bullet v = v \bullet u \\ & (tu) \bullet v = t(u \bullet v) = u \bullet (tv) \\ & (u + v) \bullet w = u \bullet w + v \bullet w \ , \quad u \bullet (v + w) = u \bullet v + u \bullet w \\ & |u| = \sqrt{u \bullet u} \\ & \theta(u, v) = \cos^{-1} \frac{u \bullet v}{|u|\,|v|} \text{ when } u, v \neq 0 \end{aligned}$$

Notice that for $u, v \neq 0$, we have $u \perp v \iff \theta(u, v) = 90^\circ \iff u \bullet v = 0$. The cross product has the flowing properties

$$\begin{aligned} & u \times v = -v \times u \\ & (tu) \times v = t(u \times v) = u \times (tv) \\ & (u + v) \times w = u \times w + v \times w \ , \quad u \times (v + w) = u \times v + u \times w \\ & \sin\theta(u, v) = \frac{|u \times v|}{|u|\,|v|} \text{ when } u, v \neq 0 \end{aligned}$$

When $u \times v \neq 0$, that is when $u, v \neq 0$ and $\sin\theta(u, v) \neq 0$ so $\theta(u, v) \neq 0^\circ$ or 180°, the vector $u \times v$ is perpendicular to u and v and points in the direction of the thumb of the right hand when the fingers curl from u to v, and the length $|u \times v|$ is equal to the area of the parallelogram on u and v (that is the parallelogram with vertices at $0, u, v$ and $u + v$).

Problems for Discussion

1: Show that the medians in any tetrahedron intersect at the point which lies $\frac{3}{4}$ of the way along each median.

2: Find the volume, and find the radius of the inscribed and the circumscribed spheres, of the regular tetrahedron with sides of length 1.

3: Explain why there are only five regular polyhedra.

4: Show that regular tetrahedra cannot be used to tile space.

5: Find coordinates of the vertices of a regular icosahedron.

6: Let u, v and w be vectors in $\mathbf{R}^3$. Show that the volume of the parallelepiped on u, v and w (that is the parallelepiped with vertices at 0, u, v, w, $u+v$, $u+w$, $v+w$ and $u+v+w$) is equal to $V = \left|(u \times v) \cdot w\right|$.

Exercises for Lesson 1

1: Let $u = (1,4,2)$ and $v = (2,5,3)$. Find $4u - 3v$, $|u|$, $u \cdot v$ and $u \times v$.

2: Let $u = (3,2,5)$, $v = (1,4,-3)$ and $w = (-3,-1,-1)$. Find the length of each side and find the tangent of each angle in the triangle with vertices at u, v and w.

3: Let $u = (1,4,2)$, $v = (2,5,3)$ and $w = (1,3,1)$. Find the volume of the parallelepiped on u, v and w, and find the volume of the tetrahedron with vertices at 0, u, v and w.

4: Find positive real numbers x, y and z such that the points $a = (2,0,0)$, $b = (-x,y,0)$, $c = (-x,-y,0)$ and $d = (0,0,z)$ are the vertices of a regular tetrahedron.

5: Find the volume, and find the radius of the inscribed and the circumscribed spheres, in the regular octahedron with sides of length 1.

6: Use coordinates of the vertices of a regular icosahedron to show that regular dodecahedra cannot be used to tile space.

7: Let u, v and w be vectors in $\mathbf{R}^3$. Show that the volume of the tetrahedron with vertices at 0, u, v and w is equal to $V = \frac{1}{6}\left|(u \times v) \cdot w\right|$.

8: Let u, v, w and z be vectors in $\mathbf{R}^3$. Show that $(u \times v) \times w = (u \cdot w)\, v - (v \cdot w)\, u$ and that $(u \times v) \cdot (w \times z) = (u \cdot w)(v \cdot z) - (u \cdot z)(v \cdot w)$.

Assorted Problems

1: Find the smallest positive integer n such that $6n$ is a square and $10n$ is a cube.

2: Find the positive integer n such that $\dfrac{1}{\log_2(2)}+\dfrac{1}{\log_3(2)}+\dfrac{1}{\log_4(2)}+\cdots+\dfrac{1}{\log_{100}(2)}=\dfrac{1}{\log_n(2)}$.

3: Find the positive integer n such that $\lfloor \log_2 1\rfloor+\lfloor \log_2 2\rfloor+\lfloor \log_2 3\rfloor+\cdots+\lfloor \log_2 n\rfloor=1300$ where, for $x\in\mathbf{R}$, $\lfloor x\rfloor$ denotes the largest integer k with $k\le x$.

4: Given a positive integer n, find the value of the sum $\frac{1}{f(1)}+\frac{1}{f(2)}+\frac{1}{f(3)}+\cdots+\frac{1}{f(n^2+n)}$ where, for a positive integer k, $f(k)$ is equal to the nearest integer to $\sqrt{k}$.

5: Find the number of integers n with $1\le n\le 2000$ such that $n=\lfloor x\rfloor+\lfloor 2x\rfloor+\lfloor 3x\rfloor+\lfloor 4x\rfloor$ for some $x\in\mathbf{R}$.

6: Find the sum of all the digits of all of the numbers $1,2,3,\cdots,2000$.

7: Find a 6-digit number $n=a_1a_2a_3a_4a_5a_6$ such that $6n=a_4a_5a_6a_1a_2a_3$.

8: Given $0<x<1$ let $A(x)$ be the area of the region bounded by the line through $(0,0)$ and $(1,1-x)$, the line through $(0,x)$ and $(1,1)$, the line through $(0,1)$ and $(1-x,0)$, and the line through $(x,1)$ and $(1,0)$. Find the value of x such that $A(x)=\frac{1}{25}$.

9: Let $\{a_n\}_{n\ge1}$ be a sequence of positive integers. Let $b_1=a_1$, $b_2=a_2b_1+1$, and for $n\ge 3$ let $b_n=a_nb_{n-1}+b_{n-2}$. Show that there does not exist $k\ge1$ such that b_k and b_{k+1} are both even.

10: Show that for every integer $n\ge 2$ there exist distinct positive integers a and b such that $\frac{1}{a}+\frac{1}{b}=\frac{1}{n}$.

11: Find every integer n with $1\le n\le 2000$ such that the decimal expansions of $\frac{1}{n}$ and of $\frac{1}{n+3}$ both terminate.

12: Show that every positive integer has a multiple of the form $11\cdots100\cdots0$.