# Math Circles, Solid Geometry

## Lesson 2: Spherical Geometry

The **unit sphere** centred at the origin is the set $S = \left\{ z \in \mathbf{R}^3 \middle| |z| = 1 \right\}$.

The **distance** between two points $u, v \in S$ is $d(u,v) = \theta(u,v) = \cos^{-1}(u \bullet v)$.

The **area** of a slice of the sphere which lies between two parallel planes separated by a distance $l$ is equal to $A = 2\pi\, l$. In particular, the area of the entire sphere is $A = 4\pi$.

The **volume** of a spherical-based cone, whose base is a region of area $A$ on the surface of the sphere and whose vertex is at the centre of the sphere, is equal to $V = \frac{1}{3}A$. In particular, the volume of the entire sphere is $V = \frac{4}{3}\pi$.

For $u \in S$ and $0 < r < \pi$, the **circle** on $S$ centred at $u$ of radius $r$ is the set $C = C(u,r) = \left\{ z \in S \middle| d(u,z) = r \right\}$. This is equal to $S \cap P$ where $P$ is the plane in $\mathbf{R}^3$ through the point $\cos(r)\, u$ perpendicular to the vector $u$.

For $u \in S$, the **line** on $S$ with **pole** $u$ is the set $L = L(u) = C(u, \frac{\pi}{2}) = \left\{ z \in S \middle| d(u,z) = \frac{\pi}{2} \right\}$. This is equal to $S \cap P$ where $P$ is the plane in $\mathbf{R}^3$ through the origin perpendicular to $u$.

The **circumference** of the circle $C(u,r)$ is equal to $L = 2\pi \sin(r)$.

The **area** of the circle $C(u,r)$ is equal to $A = 2\pi(1 - \cos(r))$.

For the **triangle** $T$ with vertices at $u, v, w \in \mathbf{S}$, we shall let $\alpha$, $\beta$ and $\gamma$ be the angles at $u$, $v$ and $w$, and we shall let $a$, $b$ and $c$ be the lengths of the sides opposite $u$, $v$ and $w$.

The **lengths** of the sides are given by $\cos a = v \bullet w$, $\cos b = w \bullet u$ and $\cos c = u \bullet v$, or alternatively by $\sin a = |v \times w|$, $\sin b = |w \times u|$ and $\sin c = |u \times v|$.

The **area** of the triangle is $A = (\alpha + \beta + \gamma) - \pi$.

The **tangent vector** at $u$ from $u$ to $v$ is the vector $u_v = \frac{(u \times v) \times u}{|u \times v|} = \frac{v - (u \bullet v)u}{|u \times v|}$. Similarly, we have the tangent vectors $u_w$, $v_u$, $v_w$, $w_u$ and $w_v$.

The **angles** of $T$ are given by $\alpha = \theta(u_v, u_w)$, $\beta = \theta(v_w, v_u)$ and $\gamma = \theta(w_u, w_v)$.

The **polar triangle** of $T$ is the triangle with vertices $u' = \frac{v \times w}{|v \times w|}$, $v' = \frac{w \times u}{|w \times u|}$ and $w' = \frac{u \times v}{|u \times v|}$. When $(u \times v) \bullet w > 0$ we have $u'' = u$, $v'' = v$, and $w'' = w$, and we have $a' = \pi - \alpha$, $b' = \pi - \beta$ and $c' = \pi - \gamma$, and we have $\alpha' = \pi - a$, $\beta' = \pi - b$ and $\gamma' = \pi - c$.

The Sine Law, The First Law of Cosines, and the Second Law of Cosines are as follows

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c},$$

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \quad \cos \beta = \frac{\cos b = \cos c \cos a}{\sin c \sin a}, \quad \cos \gamma = \frac{\cos c - \cos a \cos b}{\sin a \sin b},$$

$$\cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}, \quad \cos b = \frac{\cos \beta + \cos \gamma \cos \alpha}{\sin \gamma \sin \alpha}, \quad \cos c = \frac{\cos \gamma + \cos \alpha \cos \beta}{\sin \alpha \sin \beta}.$$

All formulas in this lesson can be modified so that they apply to a sphere of radius $R$.

## Problems for Discussion

**1:** Let $R$ be the radius of the Earth, in meters ($R \cong 6,370,000$). We describe the position of a point on the Earth in terms of its longitude $\theta$ and its latitude $\phi$. Find the distance (expressed as a multiple of $R$) and the bearing (expressed as an angle north of east) from the point at $(\theta,\phi)=(\frac{\pi}{3},\frac{\pi}{6})$ to the point at $(\theta,\phi)=(\frac{\pi}{2},\frac{\pi}{4})$.

**2:** Some people live on a spherical planet. They carefully survey a large triangle and find that its angles are $90.0000°$, $45.0000°$ and $45.0126°$, and that its area is $22,000\ \text{m}^2$. Using a calculator, determine the radius of the planet.

**3:** Consider the square with vertices at $(1,\pm1,0)$, $(1,0,\pm1)$ on the sphere of radius 2 centred at the origin. Find the area of the region on the sphere that lies outside the square but inside its circumcircle.

## Exercises for Lesson 2

**1:** Find $w\in S$ such that the line $L(w)$ passes through $u=\frac{1}{\sqrt{5}}(0,2,1)$ and $v=\frac{1}{\sqrt{3}}(1,1,-1)$.

**2:** Let $u=\frac{1}{\sqrt{2}}(1,1,0)$ and $v=\frac{1}{\sqrt{2}}(0,1,1)$. Find the points of intersection of $L(u)$ and $L(v)$.

**3:** Let $u=(0,-1,0)$, $v=\frac{1}{\sqrt{2}}(1,1,0)$ and $w=\frac{1}{\sqrt{3}}(1,1,1)$. Find the vectors $u_v$ and $u_w$ and find the angle $\alpha$ at the vertex $u$ in the triangle with vertices $u,v,w\in S$.

**4:** Find the area and perimeter of a triangle on $S$ with interior angles $\frac{\pi}{4},\frac{\pi}{3},\frac{\pi}{2}$.

**5:** Find the perimeter of a regular hexagon on $S$ with interior angles equal to $\frac{5\pi}{6}$.

**6:** Find the area of a regular hexagon on $S$ with sides of length $\frac{\pi}{6}$.

**7:** Let $u=\frac{1}{\sqrt{2}}(1,-1,0)$, $v=\frac{1}{\sqrt{2}}(1,1,0)$ and $w=\frac{1}{\sqrt{2}}(0,1,1)$. Find the area of the triangle $T$ on the sphere $S$ given by $T=\left\{z\in S \,\middle|\, d(z,u)\leq\frac{\pi}{2},\ d(z,v)\leq\frac{\pi}{2} \text{ and } d(z,w)\leq\frac{\pi}{2}\right\}$.

**8:** Let $u=\frac{1}{\sqrt{6}}(2,1,-1)$, $v=\frac{1}{\sqrt{6}}(1,2,1)$ and $w=\frac{1}{\sqrt{6}}(1,-1,2)$. Find the circumcenter of the triangle $T$ with vertices at $u,v,w\in S$.

**9:** A light is at a distance of 5 units from the centre of a sphere of radius 3 units. Find the area of the portion of the surface of the sphere which is lit.

**10:** Find the radius $R$ of a sphere on which there is a regular (equilateral) triangle with sides of length 1 and angles equal to $\frac{2\pi}{5}$.

Assorted Problems

**1:** Show that every integer $n > 11$ is the sum of two composite numbers.

**2:** A pile of potatoes initially weighs 100 kg and the potatoes are 99% water, by weight. The pile is left to dry until the potatoes are 98% water. How much does the pile now weigh?

**3:** A rectangular piece of paper is folded once so that two diagonally opposite corners coincide. The length of the crease is equal to the length of the longer side of the rectangle. Find the ratio of the long side to the short side.

**4:** Let $n$ be a positive integer. Find the number of ordered pairs of integers $(x, y)$ such that $|x| + |y| < n$.

**5:** Find three consecutive even numbers whose product is an 8-digit number that begins with the digits 87 and ends with the digit 8.

**6:** Find the sum of all of the 4-digit even numbers which use only the digits 0, 1, 2, 3, 4, 5 (digits can be repeated).

**7:** Let $a_1 = 1$ and $a_2 = 2$ and for $n \geq 3$ let $a_n = \dfrac{1}{(a_{n-1})^{\phi}(a_{n-2})}$ where $\phi = \frac{1+\sqrt{5}}{2}$. Find $a_{1000}$.

**8:** Let $a_1 = 1$ and for $n \geq 2$ let $a_n = n^{a_{n-1}}$. Find the last two digits of $a_{16}$.

**9:** Show that $\lfloor (2+\sqrt{3})^n \rfloor$ is odd for every positive integer $n$.

**10:** Find all real numbers $x$ such that $x \Big\lfloor x \lfloor x \lfloor x \rfloor \rfloor \Big\rfloor = 88$.

**11:** A point $P$ is chosen at random inside an equilateral triangle $ABC$. Find the probability that one of the triangles $ABP$, $APC$ and $PBC$ is acute-angled.

**12:** Let $S = \{1, 2, 3, \cdots, 10\}$. Show that for every sequence $a_1, a_2, \cdots, a_{20}$ with each $a_i \in S$, there exist $n, m$ with $1 \leq n \leq m \leq 20$ such that the product $a_n a_{n+1} \cdots a_m$ is a square.