WATERLOO
MATHEMATICS
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Senior Math Circles: Geometry III

Review of Important Facts About Trigonometry

- Most famous trig identity: $\sin^2 x + \cos^2 x = 1$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\sin(x - y) = \sin x \cos y - \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$
- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x$
- $\tan(2x) = \dfrac{2 \tan x}{1 - \tan^2 x}$
- $\sin(-x) = -\sin x$
- $\cos(-x) = -\cos x$
- $\tan(-x) = -\tan x$

Practicing with these facts

1. Calculate $\sin(105°)$.
2. Calculate $\cos(22.5°)$.
3. Simplify $\cos(90° + x)$.
4. Simplify $\sin(180° - x)$.
5. Simplify $\cos(90° - x)$.

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Problem Set 2

1. If $0^\circ < x < 90^\circ$ and $\tan(2x) = -\frac{24}{7}$, determine the value of $\sin x$.
(Source: 2002 Descartes Contest)

2. Determine the values of x, $0 < x < \pi$, for which $\dfrac{1}{2+\cos^2 x} = \dfrac{4}{11}$.
(Source: 2001 Descartes Contest)

3. (a) Prove that $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$, where $0 < A < \frac{\pi}{2}$.
(b) If $\sin 2A = \frac{4}{5}$, find $\tan A$.
(Source: 1998 Descartes Contest)

4. For what values of θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, does the equation $x^2 + (2\sin\theta)x + \cos 2\theta = 0$ have real roots?
(Source: 1996 Descartes Contest)

5. (a) If $\tan x = a$, $0 < x < \frac{\pi}{2}$, find $\sin 2x$ in terms of a.
(b) Given the equations

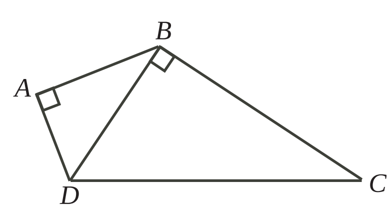
$$\begin{aligned} \cos x + \cos y &= 2m \\ (\cos x)(\cos y) &= -3m^2 \end{aligned}$$

where $m \in \mathbb{R}$, find an expression for $\sin^2 x + \sin^2 y$ in terms of m.
(Source: 2000 Descartes Contest)

6. Determine the points of intersection of the curves defined by $y = 8\cos x + 5\tan x$ and $y = 5\sec x + 2\sin 2x$ for $0 \le x \le 2\pi$.
(Source: 1997 Descartes Contest)

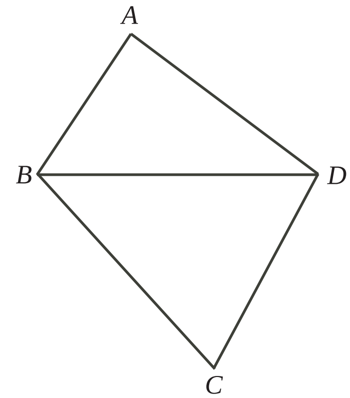
7. Prove that there are no real values of x such that $2\sin x = x^2 - 4x + 6$.
(Source: 1993 Euclid Contest)

8. In the quadrilateral $ABCD$, angles DBC and DAB are right angles. Also, $\angle ADB = \angle BDC = 60^\circ$ and $DC = 4$. Determine which is greater: $DA + AC$ or $DB + BC$.
(Source: 1997 Descartes Contest)



9. $\triangle ABC$ has $\angle ABC = 30°$, $AB = 150$, and $AC = 50\sqrt{3}$. Determine the length of BC.
(Source: 1976 Euclid Contest)

10. (a) The quadrilateral $ABCD$ has $AB = 5$, $BC = 6\sqrt{2}$ and $AD = 7$. If $BD = 8$ and $\angle ABC = 105°$, determine the length of CD.



(b) Prove the identity: $\sin(A - B)\sin(A + B) = \sin^2 A - \sin^2 B$.

(Source: 1999 Descartes Contest)

11. In $\triangle ABC$, $a = 3\sqrt{2}$, $b = 4\sqrt{2}$, $\angle BAC = 45°$, and $\angle ABC$ is obtuse. Determine c.
(Source: 1977 Euclid Contest)

12. (a) Determine the constants a and b so that $\dfrac{-3 + 4\cos^2\theta}{1 - 2\sin\theta}$ is equal to $a + b\sin\theta$ for all values of θ.

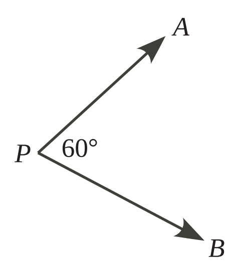
(b) Find all values of x, $0 \le x \le 2\pi$, which satisfy $\sin^2 x + \cos x + 1 = 0$.

(Source: 1977 Euclid Contest)

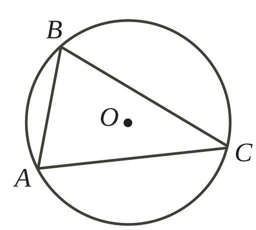
13. Find all angles x in the interval $-\pi \le x \le \pi$ such that $\sin 2x + \sin 3x = \sin x$.

14. In $\triangle ABC$, $\sin B = \frac{3}{5}$ and $\sin C = \frac{1}{4}$. What is the ratio $AB : BC$?
(Source: 1978 Euclid Contest)

15. Two ships, A and B leave port P at the same time, travelling at constant speeds of 20 km/h and 32 km/h, respectively. If the angle separating their paths is 60°, what is the distance between their positions after 2.5 hours?
(Source: 1998 Descartes Contest)



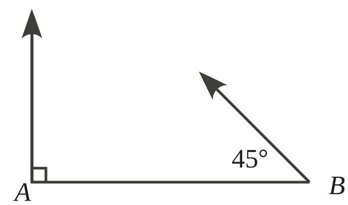
16. $\triangle ABC$ has its vertices on a circle with radius 2 and its centre at O, as shown. If $AC = 3$, calculate the cosine of $\angle ABC$.
(Source: 2000 Descartes Contest)



17. In triangle ABC, given that $b\cos A = a\cos B$, prove that $a = b$.
(Source: 1994 Euclid Contest)

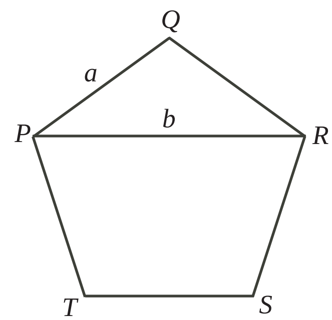
18. The length of the hands of a clock are 6 cm and 4 cm, respectively. What is the distance between the tips of the hands at two o'clock?
(Source: 1978 Euclid Contest)

19. Starting at 10:00 a.m. from A, a runner runs due north at a speed of 10 km/h. Starting 30 minutes later from B, which is 25 km east of A, a cyclist travels northwest at a constant speed. The runner and the cyclist arrive at the same point at the same time. Determine the speed of the bicycle.
(Source: 1996 Euclid Contest)



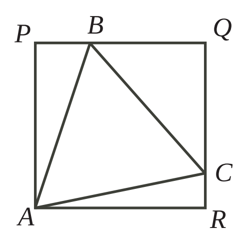
20. In $\triangle ABC$, angle A is twice angle B. Prove that $a^2 = b(b + c)$.

21. Let a be the length of a side and b the length of a diagonal in the regular pentagon $PQRST$. Prove that $\frac{b}{a} - \frac{a}{b} = 1$.
(Source: 1998 Descartes Contest)



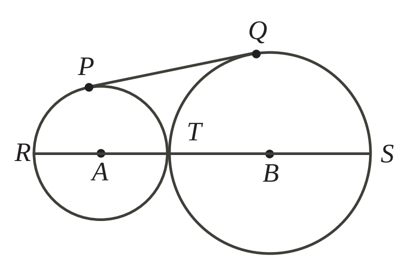
22. A regular octagon, $ABCDEFGH$, is inscribed in a circle of radius 1. Prove that the product of the lengths of the line segments joining A to each of the other vertices equals 8.
(Source: 1979 Euclid Contest)

23. An equilateral triangle ABC of side length 1 is inscribed in the rectangle $APQR$ so that B lies on PQ and C lies on QR, as shown. Prove that the area of $\triangle BQC$ is equal to the sum of the areas of $\triangle APB$ and $\triangle ARC$.
(Source: 2002 Descartes Contest)

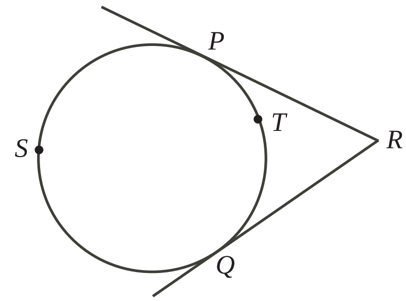


24. Two circles of radii 4 and 2 have their centres 4 units apart and intersect at X and Y. A line drawn through X cuts the circles at A and B, and AXB meets the line of centres produced at an angle of 30°. Calculate the length of AXB.
(Source: 1977 Euclid Contest)

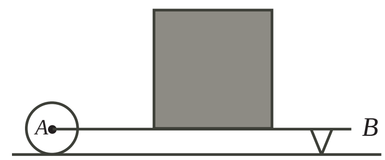
25. In the diagram, the circles with centres A and B are tangent externally at T. PQ is a common tangent line. The line of centres also intersects the circles at R and S, as shown. RP and SQ, when produced, meet at X. Prove that $\angle RXS$ is a right angle.
(Source: 1978 Euclid Contest)



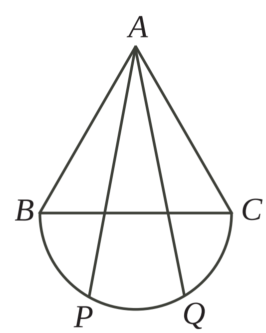
26. PR and QR are tangents to the given circle. If arc $PSQ = 4$ arc PTQ, what is $\angle PRQ$ in degrees?
(Source: 1977 Euclid Contest)



27. In the diagram, a box in the shape of a cube with side 1.2 m is sitting on a hand cart 1 m from the front end, A, of the cart platform. The platform AB is parallel to the floor. The wheels on the cart have radii 0.4 m. The back end of the platform is lifted so that the point AB is rotated 15° about the point A. What is the minimum height of a doorway through which the cart can pass?
(Source: 1996 Euclid Contest)



28. For the equilateral triangle ABC, side BC is the diameter of a semicircle. Points P and Q create equal arcs $BP = PQ = QC$ on the semicircle. Show that line segments AP and AQ trisect side BC.
(Source: 1995 Euclid Contest)



29. In $\triangle ABC$, $\angle C = \angle A + 60°$. If $BC = 1$, $AC = r$, and $AB = r^2$, where $r > 1$, prove that $r < \sqrt{2}$.
(Source: 1996 Euclid Contest)

30. In $\triangle ABC$, angle A is acute and P is any point on BC. The reflection of point P in AC is point T, and the reflection of point P in AB is point S. Determine the position of point P so that the area of $\triangle ATS$ is a minimum.
(Source: 1992 Euclid Contest)

31. The hypotenuse BC is right triangle ABC is divided into three equal parts by lines AQ and AP. If $AQ = 9$ and $AP = 7$, what is the length of BC?

For the remaining problems, let A, B and C be the angles of a triangle.

32. Prove that $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \sin A + \sin B + \sin C$.
33. Prove that $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \cos A + \cos B + \cos C - 1$.
34. Prove that $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$.
35. Prove that $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.
36. Prove that $(1 + \tan A)(1 + \tan B) = 2$ if and only if $A + B = \frac{\pi}{4}$.
37. Prove that $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.
38. Prove that $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$.