WATERLOO
MATHEMATICS
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Senior Math Circles: Geometry I

Opening Problem

(a) If $\frac{30}{7} = x + \cfrac{1}{y + \cfrac{1}{z}}$, where x, y and z are positive integers, then what is the value of $x + y + z$?

(b) Determine the value of $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$.

(c) Determine the value of $3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \ddots}}}}$

(e) Determine the value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{\cdots}}}}$.

(f) If $a > 0$, determine the value of $\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{\cdots}}}}$.

(g) Determine all real numbers $a > 0$ for which $\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{\cdots}}}} = a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \ddots}}}$.

(d) CHALLENGE
Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \ldots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers. (Source: 1995 Putnam Competition)

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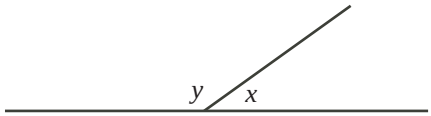
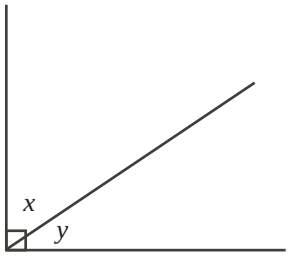
Facts About Angles

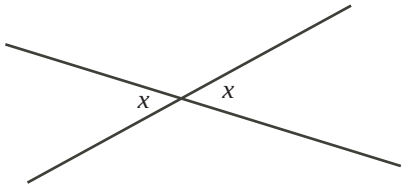
AF1. Angles along a straight line add to 180°.
They are called *supplementary* angles.



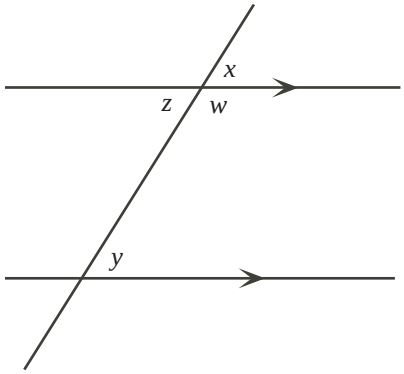
AF2. Angles in a right angle add to 90°.
They are called *complementary* angles.



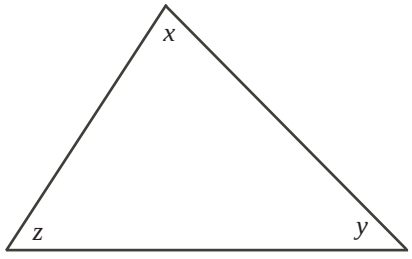
AF3. Opposite angles are equal. Why?
Angles around a point add to 360°. Why?



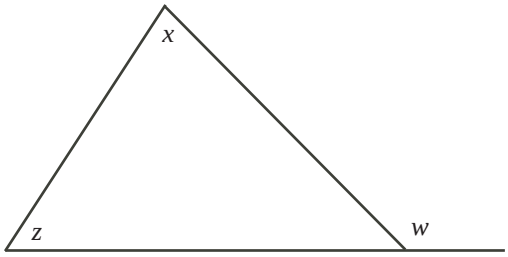
AF4. Consider two parallel lines.
Corresponding angles are equal ($x = y$).
Alternating angles are equal ($y = z$).
Interior angles are supplementary ($w + y = 180°$).



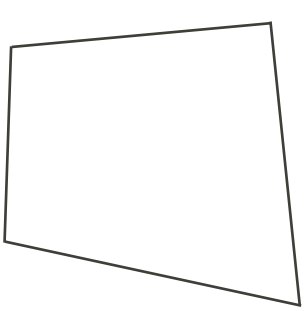
AF5. Angles in a triangle add to 180°.
Can you prove this?

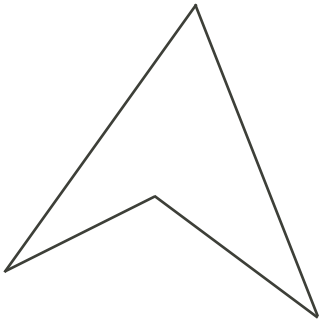


AF6. The exterior angle of a triangle equals the sum of the other two interior angles. That is, $w = x + z$.



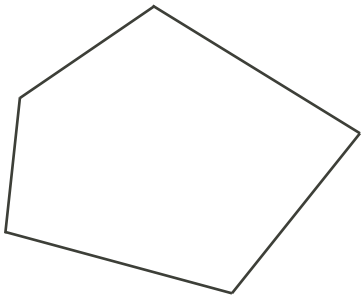
AF7. What is the sum of the angles in a quadrilateral?
How could we figure this out?



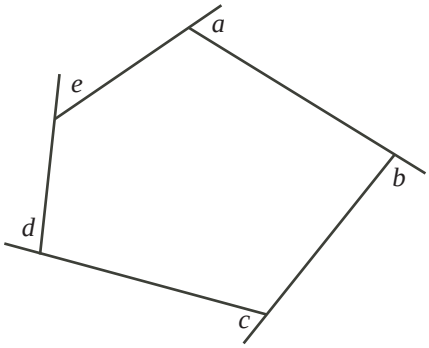


AF8. What is the sum of the angles in a pentagon?
In a hexagon?
In a polygon with n sides?

A *regular* polygon has all angles equal and all side lengths equal.
What is each angle in a regular pentagon?
What is each angle in a regular hexagon?
What is each angle in a regular decagon?

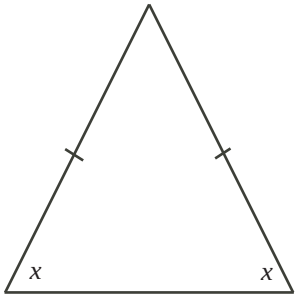


AF9. The sum of the exterior angles in a pentagon is 360°.
Can you find two different ways to show this?
The sum of the exterior angles in any convex polygon is 360°.



AF10. A triangle is called *isosceles* if two of its side lengths are equal and is called *equilateral* if all three side lengths are equal.

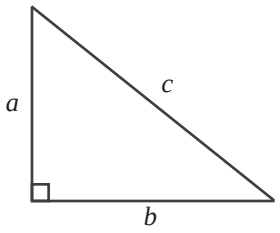
If a triangle has two side lengths equal, then the opposite two angles are equal.



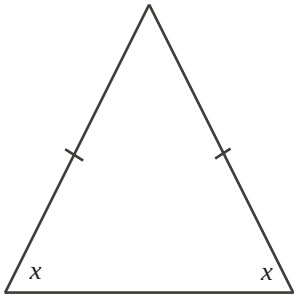
Facts About Side Lengths

SF1. In a right-angled triangle, the Pythagorean Theorem tells us that $a^2 + b^2 = c^2$.
Can you prove this?

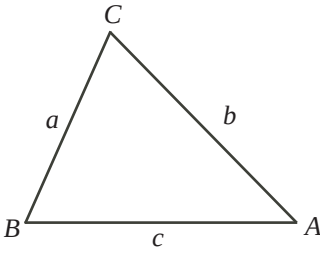
A *Pythagorean Triple* is a triple (a, b, c) of positive integers with $a^2 + b^2 = c^2$. What Pythagorean Triples do you know?



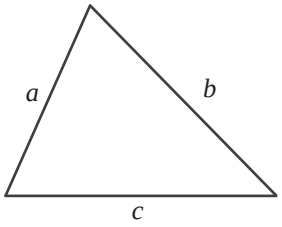
SF2. If a triangle has two angles equal, then the two opposite sides are equal.



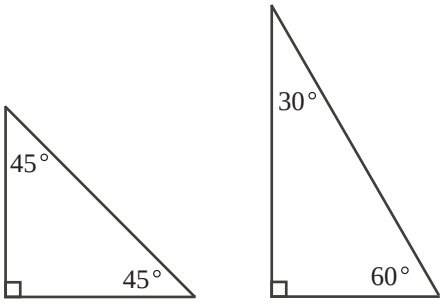
SF3. If $\angle A < \angle B$, then $a < b$.
If $a < b$, then $\angle A < \angle B$.



SF4. If a, b and c are the side lengths of a triangle, the Triangle Inequality tells us that $b + c > a$ and $a + c > b$ and $a + b > c$.
Can you explain why this is true?

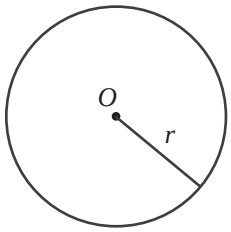


SF5. There are two kinds of special triangles.
The first has angles 45°, 45° and 90°.
The second has angles 30°, 60° and 90°.
If the shortest side in each has length 1, what are the other side lengths?
These can be scaled by any factor.

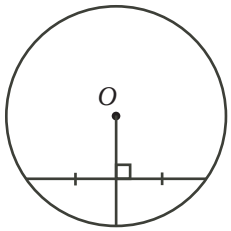


Facts About Circles

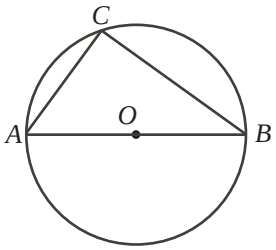
CF1. A circle is determined by its centre and its radius.
A circle is the set of points at a fixed distance from a fixed point.



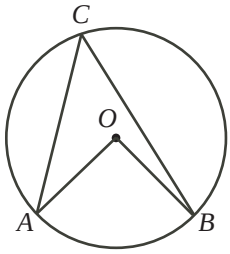
CF2. A chord is a line segment that begins and ends on a circle.
If a radius cuts a chord into two equal parts, then the radius is perpendicular to the chord.
If a radius is perpendicular to a chord, then it cuts the chord into two equal parts.



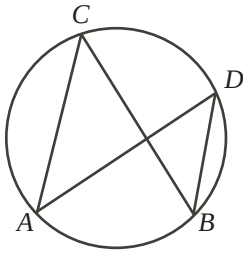
CF3. If AB is a diameter, then $\angle ACB = 90°$.



CF4. If AB is a chord and C is on the major arc, then $\angle AOB = 2\angle ACB$.



CF5. If AB is a chord and C and D are on the major arc, then $\angle ACB = \angle ADB$.



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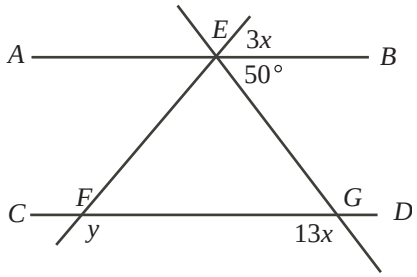
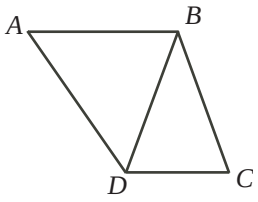
Problem Set

1. In the diagram, AB is parallel to CD. Determine the values of x and y.

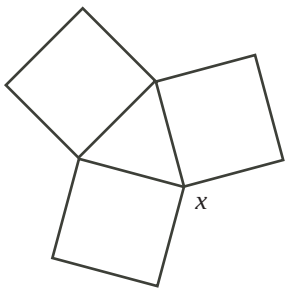


2. Triangle ABC has a right angle at B. AC is extended to D so that $CD = CB$. The bisector of angle A meets BD at E. Prove that $\angle AEB = 45°$.

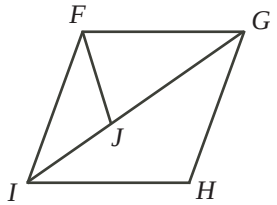
3. In the diagram, AB is parallel to DC and $AB = BD = BC$. If $\angle A = 52°$, determine the measure of $\angle DBC$.



4. The diagram shows three squares of the same size. What is the value of x?

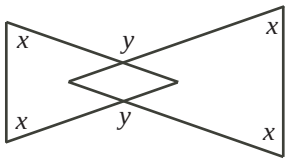


5. The diagram shows a rhombus $FGHI$ and an isosceles triangle FGJ in which $GF = GJ$. Angle FJI equals $111°$. What is the measure of angle JFI?

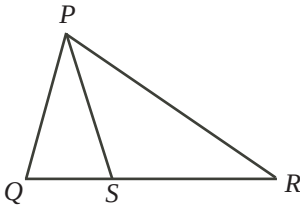


6. $ABCD$ is a square. The point E is outside the square so that CDE is an equilateral triangle. Find angle BED.

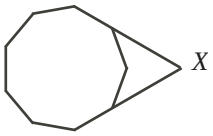
7. The diagram shows two isosceles triangles in which the four angles marked x are equal. The two angles marked y are also equal. Find an equation relating x and y.



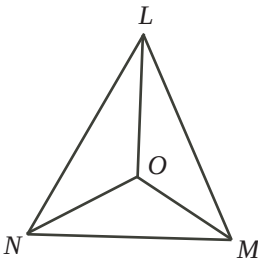
8. In the diagram, QSR is a straight line. $\angle QPS = 12°$ and $PQ = PS = RS$. What is the size of $\angle QPR$?



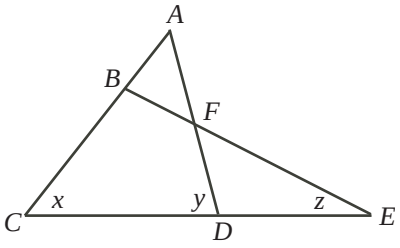
9. The diagram shows a regular nonagon with two sides extended to meet at point X. What is the size of the acute angle at X?



10. The three angle bisectors of triangle LMN meet at a point O as shown. Angle LNM is 68°. What is the size of angle LOM?



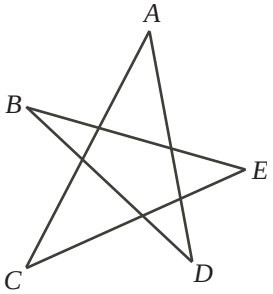
11. In the figure shown, $AB = AF$ and ABC, AFD, BFE, and CDE are all straight lines. Determine an equation relating x, y and z.



12. The angles of a nonagon are nine consecutive numbers. What are these numbers?

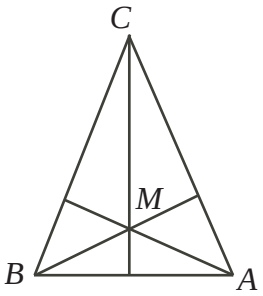
13. What is the measure of the angle formed by the hands of a clock at 9:10?

14. Determine the sum of the angles A, B, C, D, and E in the five-pointed star shown.

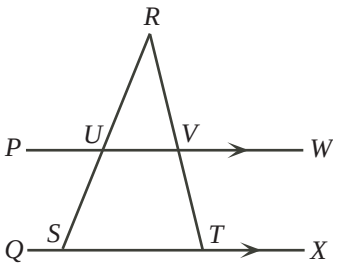


15. In $\triangle PQR$, $PQ = PR$. PQ is extended to S so that $QS = QR$. Prove that $\angle PRS = 3(\angle QSR)$.

16. In the diagram, let M be the point of intersection of the three altitudes of triangle ABC. If $AB = CM$, then what is $\angle BCA$ in degrees?

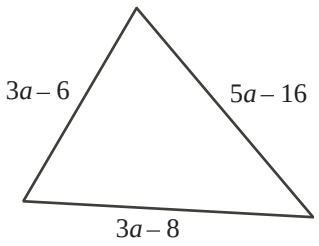


17. In the diagram, PW is parallel to QX, S and T lie on QX, and U and V are the points of intersection of PW with SR and TR, respectively. If $\angle SUV = 120°$ and $\angle VTX = 112°$, what is the measure of $\angle URV$?

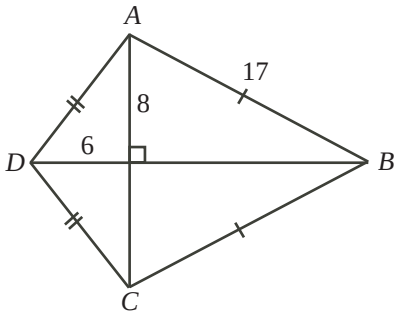


18. Three regular polygons meet at a point and do not overlap. One has 3 sides and one has 42 sides. How many sides does the third polygon have? Can you find other sets of three polygons that have this property?

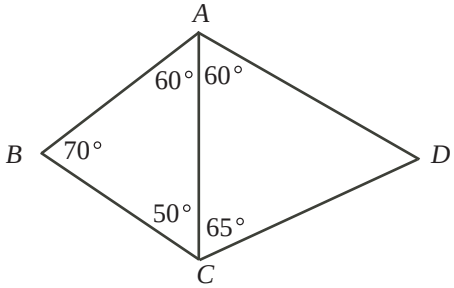
19. Determine the number of different values of a for which the given triangle is isosceles.



20. Calculate the area of figure $ABCD$.



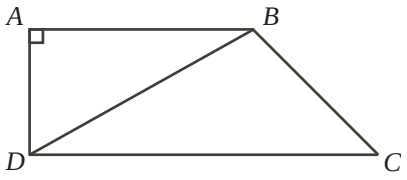
21. In the diagram, which side is the longest: AB, BC, AC, CD, or AD?



22. If a 3 m stake casts a shadow 7 m long, what is the height of a tree that casts a shadow 63 m long?

23. A *scalene* triangle is a triangle whose side lengths are all different. Determine the side lengths of all possible scalene triangles with integer side lengths and perimeter less than 13.

24. In the diagram, $AB = 4$, $DC = 6$, and AB is parallel to DC. If $\angle C = 45°$, determine the length of BD.

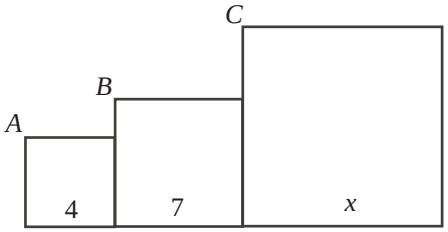


25. Triangle ABC is similar to $\triangle XYZ$. If $AB = 4$, $YZ = 9$, and $BC = XY = p$, determine the value of p.

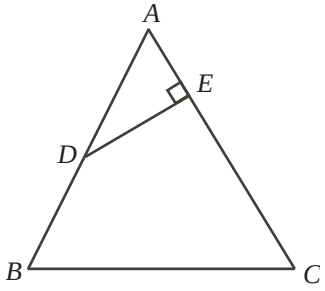
26. A triangle can be formed having side lengths 4, 5 and 8. It is impossible however, to construct a triangle with side lengths 4, 5 and 10. Using the side lengths 2, 3, 5, 7 and 11, how many different triangles *with exactly two equal sides* can be formed?

27. A triangle can be formed having side lengths 4, 5 and 8. It is impossible however, to construct a triangle with side lengths 4, 5 and 10. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

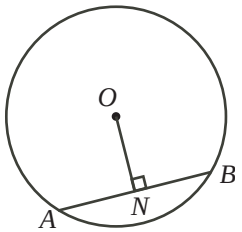
28. In the adjacent squares shown, the vertices A, B and C lie in a straight line. What is the value of x?



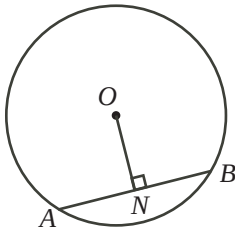
29. In the diagram, $AD = BD = 5$, $EC = 8$ and $AE = 4$. Determine the length of BC.



30. Determine the length of the chord AB if $OA = 5$ and $ON = 3$.

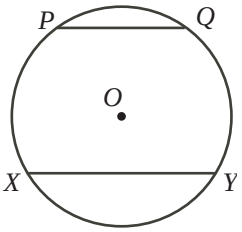


31. If $AB = 10$ and $OA = 13$, determine the length of ON.

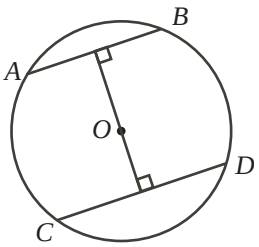


32. A circle has a diameter of length 26. If a chord of the same circle has a length of 10, how far is the chord from the centre?

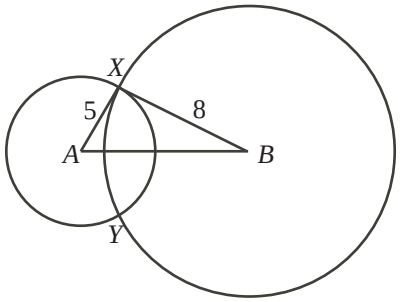
33. Calculate the distance between the parallel chords PQ and XY if $PQ = 6$, $XY = 8$, and the radius of the circle is 5.



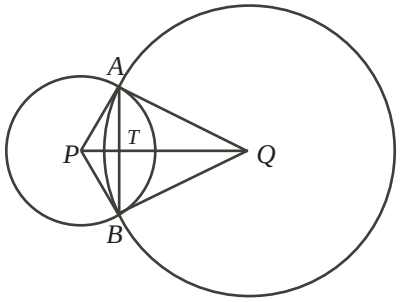
34. The two parallel chords AB and CD are a distance of 14 part. If AB has length 12 and the radius of the circle is 10, calculate the length of CD.



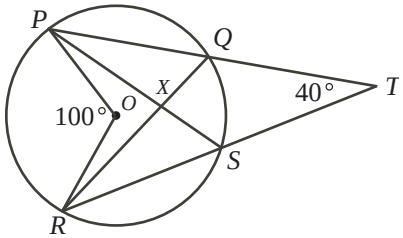
35. Two circles with centre A and B have radii 5 and 8, respectively. The circles intersect at the points X and Y. If $XY = 8$, determine the length of AB, the distance between the centres.



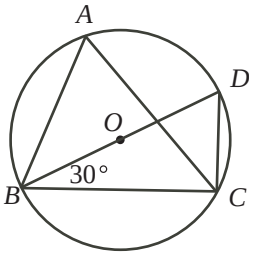
36. In the diagram, $PA = 13$ and $QA = 20$, where P and Q are the centres of the circles. Determine the length of AB if $PQ = 21$.



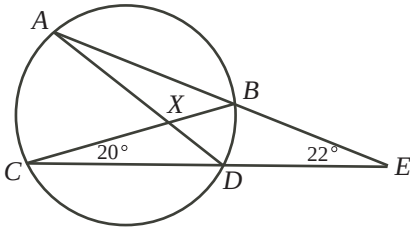
37. In the diagram, O is the centre of the circle. Determine the measure of $\angle QXS$.



38. Determine the measure of $\angle BAC$.

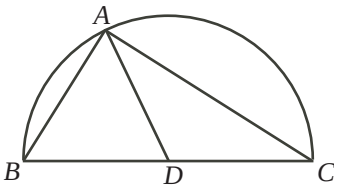


39. Determine the measure of $\angle ADC$ and of $\angle AXB$.

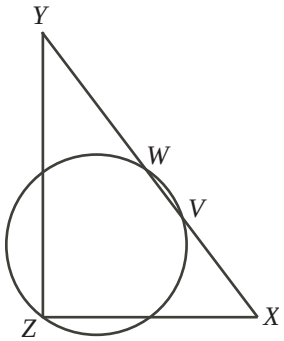


40. A cyclic quadrilateral is a quadrilateral that has all four its vertices on the same circle. Prove that opposite angles are supplementary.

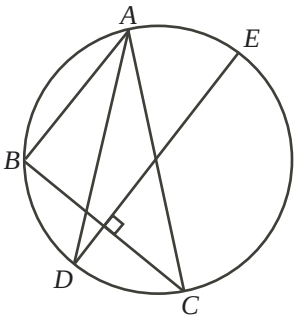
41. In the diagram, $\triangle ABC$ is inscribed in the semicircle with centre D. If $AB = AD$, determine the measure of $\angle ACD$.



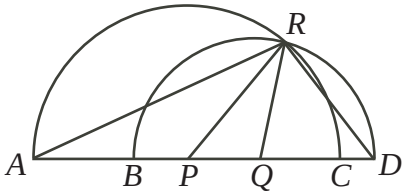
42. In the diagram, $\triangle XYZ$ is right-angled at Z. W is the midpoint of XY, and the circle with diameter ZW intersects WX at V. If $XY = 50$ and $WV = 7$, determine the length of XZ.



43. $\triangle ABC$ has its vertices on a circle, as shown. The bisector of the angle at A meets the circumference at D. From D, a line is drawn perpendicular to the chord BC so that it meets the circumference at E. Prove that DE is a diameter of the circle.



44. In the diagram, points B, P, Q, and C lie on line segment AD. The semi-circle with diameter AC has centre P and the semi-circle with diameter BD has centre Q. The two semi-circles intersect at R. If $\angle PRQ = 40°$, determine the measure of $\angle ARD$.



45. If O is the centre of the circle and $\angle BCD = 82°$, what is the value of x in degrees?