



Grade 6 Math Circles

Fall 2012

Patterns

Number Patterns

Definition: A *pattern* is a set of numbers or objects in which all the terms are related with each other by a specific rule. This rule can be a written description or a formula. Patterns are also known as sequences.

Definition: Each number in the pattern is called a *term*, and the n^{th} term is denoted as a_n .

Example

Examine the following number pattern: 2, 4, 6, 8, 10, ...

There are two ways that we could explain this pattern. First, we can say that this pattern starts with 2 and the next number is determined by adding 2. The second way is to say that it is all even positive integers.

As noted before, when dealing with patterns, we use a_n to indicate the n^{th} term in the pattern. So the term previous to the n^{th} term is a_{n-1} . Therefore, in Example 1, we could say that $a_1 = 2$, $a_2 = 4$, $a_3 = 6$ and so on.

Then what is the term a_n ? We said that each new term is found by adding 2 to the previous term, so we can define a_n to be $a_n = a_{n-1} + 2$.

This is called a *recursive* definition, meaning that each new term depends on the term before it.

Exercise 1

Describe the following patterns recursively. Then use the rule to find the 6^{th} and 7^{th} terms.

- a) 125, 130, 135, 140, 145, ...

$$a_1 = 125, a_n = a_{n-1} + 5$$

$$a_6 = a_5 + 5 = 145 + 5 = 150$$

$$a_7 = a_6 + 5 = 150 + 5 = 155$$

- b) 2, 4, 8, 16, 32, ...

$$a_1 = 2, a_n = 2a_{n-1}$$

$$a_6 = 2a_5 = 2 * 32 = 64$$

$$a_7 = 2a_6 = 2 * 64 = 128$$

c) * 2, 3, 5, 8, 12, ...

$$a_1 = 2, a_n = a_{n-1} + n - 1$$

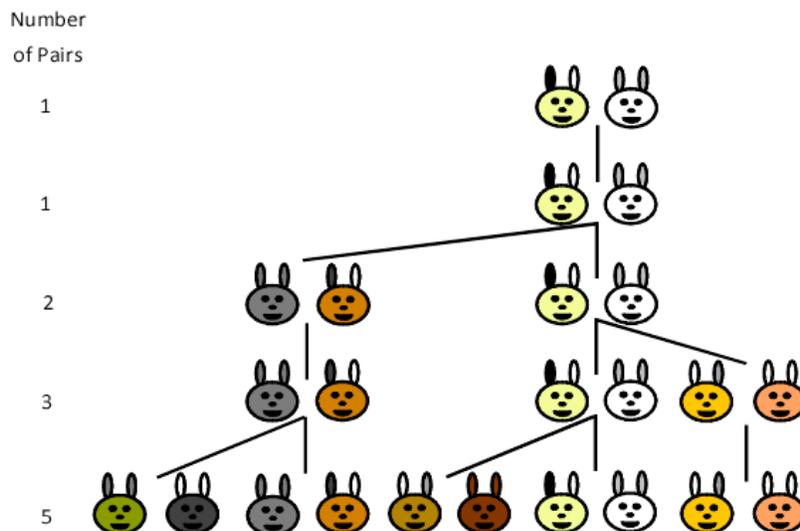
$$a_6 = a_5 + 6 - 1 = 12 + 5 = 17$$

$$a_7 = a_6 + 7 - 1 = 17 + 6 = 23$$

Fibonacci's Rabbits

In the year 1202, Leonardo Fibonacci was investigating how fast rabbits could breed. Suppose a newly-born pair of rabbits, one male, one female, are put in a field. Rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits. Suppose that our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month on. How many pairs of rabbits will there be in one year?

- At the end of the first month, the rabbits mate, but there is still only one pair.
- At the end of the second month, the female has produced a new pair, so now we have two pairs of rabbits.
- At the end of the third month, the original female has produced another pair, making three pairs of rabbits.
- At the end of the fourth month, the original female produces her third pair and the female born two months ago produces her first pair, making five pairs of rabbits.



So far, we have this pattern: 1, 1, 2, 3, 5, ...

Can you figure out what the pattern rule is in words? How about recursively? [Add the previous two terms together to get the next term or \$a_n = a_{n-1} + a_{n-2}\$.](#)

What are the next 4 terms of the pattern? [8, 13, 21, 34](#)

Now that you know how the pattern is formed, answer the initial question: How many pairs of rabbits will there be in one year? [144](#)

This pattern is called the *Fibonacci Sequence*. This is a very important sequence as the numbers are often found in nature: pine cones, flowers, nautilus, etc.

However, what if we want to find the 1000th term in the pattern? That would mean that we have to calculate all of the 999 previous terms before we can determine the value of the 1000th term. This would take a really long time to do.

Instead, we want to find an explicit rule that does not require any of the previous terms. If we take a look at the pattern, we can see that the n^{th} term is the n^{th} multiple of 2. Therefore, the rule is $a_n = 2n$.

Exercise 2

Describe the following patterns explicitly. Then use the rule to find the 6th and 7th terms.

a) 3, 4, 5, 6, 7, ...

$$a_n = n + 2$$

$$a_6 = 6 + 2 = 8$$

$$a_7 = 7 + 2 = 9$$

b) 3, 6, 9, 12, 15, ...

$$a_n = 3n$$

$$a_6 = 3 * 6 = 18$$

$$a_7 = 3 * 7 = 21$$

c) 1, 4, 9, 16, 25, ...

$$a_n = n * n$$

$$a_6 = 6 * 6 = 36$$

$$a_7 = 7 * 7 = 49$$

d) 2, 4, 8, 16, 32, ...

$$a_n = 2^n$$

$$a_6 = 2^6 = 64$$

$$a_7 = 2^7 = 128$$

Tower of Hanoi



Tower of Hanoi is a game where you have three pegs and three (or more) rings of different sizes stacked onto the first peg, largest on the bottom and smallest on the top. The objective is to move your stack of rings to the third peg. However, there are a few rules to follow: you may only move one ring at a time and you may never have a larger ring on top of a smaller ring.

Q: What is the smallest number of moves needed to complete the game with:

- a) one disc? 1
- b) two discs? 3
- c) three discs? 7
- d) four discs? 15

Q: Do you notice any patterns in the way the number of moves increases?

If you look back at Exercise 2 part d), you can see that the numbers in that pattern are almost the same as this pattern. The only difference is that the values are one less than the previous example. Therefore, the minimum number of moves needed to move n rings is $a_n = 2^n - 1$.

Q: There is a legend that a 64-disc version of the Tower of Hanoi is being played out in a temple, and when the final move is made, the world will come to an end. If one move is made each second (even though this is very unlikely) and there are no incorrect moves, how long would it take to complete the game with 64 discs? Do we need to worry yet, if the first disc was moved at the very beginning of time (approximately 13.75 billion years ago)?

$$a_{64} = 2^{64} - 1 = 18446744073709551616 - 1 = 18446744073709551615 \text{ seconds}$$

To find the number of years it will take, first divide by 60, then by 60, then by 24, then by 365.25. This gives us that it will take approximately 584.5 billion years to complete the 64 disc puzzle. Therefore we do not need to worry about the world ending anytime soon.

Problem Set

1. Determine the recursive rule for the patterns below. Then, using the rule, find the next two terms in each pattern.
 - a) 1, 3, 9, 27, 81, ...
 - b) 1, 5, 13, 29, ...
 - c) * 5, -5, 5, -5, 5, ...
 - d) * 2, 3, 5, 8, 12, ...

2. Determine the explicit rule for the patterns below. Then, using the rule, find the next two terms in each pattern.
 - a) 4, 8, 12, 16, 20, ...
 - b) 6, 8, 10, 12, ...
 - c) ** 1, 2, 6, 24, 120, ...
(HINT: $n! = n * (n - 1) * \dots * 3 * 2 * 1$, for example $4! = 4 * 3 * 2 * 1$)

3. Determine the missing number(s) of the following patterns:

a) $\underline{\quad}, 9, 27, 81, \underline{\quad}, 729, \dots$

b) $70, \underline{\quad}, 56, 49, \underline{\quad}, 35, \dots$

c) $415, 257, 158, 99, \underline{\quad}, 40, \underline{\quad}, \dots$

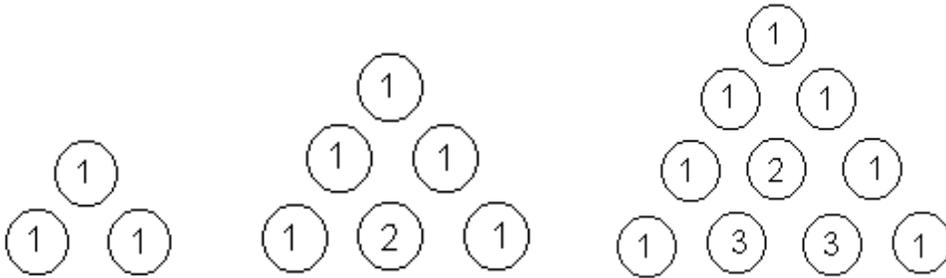
4. If the following 5 arrow pattern repeats itself continuously, what arrow would be in the 48th position?



5. Jack has a jar of magic jellybeans. The number of jellybeans doubles every day. After 16 days, the jar is full. After how many days was the jar half full?
6. What is the value of $1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \dots + 45 + 46 - 47 - 48 + 49$?
7. Five students named Fred, Jennifer, Henry, Emily and George are seated around a table in that order. They decide to play a "countdown" to determine who goes first in Monopoly. Henry starts by saying 34, then Emily says 33. If they continue to count down in this way, who will say 1?
8. Ron ate a total of 100 peanuts over a period of 5 days. Each day he ate six more peanuts than on the previous day. How many peanuts did Ron eat on the third day?
9. Jessica decided to train for a marathon, which is 26 miles. On the first day, she runs a distance of 2 miles. The next day she runs 5 miles. On the third and fourth days, she runs 8 miles and 11 miles, respectively. On what day will she run the full marathon (26 miles)?
10. In a science experiment, the scientists observed that during each 10 minute time period, $\frac{1}{4}$ of the cells present died and the remaining cells all doubled. If there were 32 cells initially, how many were there after 40 minutes?
11. Christine is trying to get across a stream while on a hike, by using a path of stones. She first jumps from the bank of the stream onto the 2nd stone. From there she jumps onto the 4th stone and then the 6th. If the stone path is 49 stones long and each of her steps takes 2 seconds, how long will it take for Christine to get to the other side of the stream?

12. Pascal's Triangle

Determine the next 4 triangles in the following pattern.



13. The following pattern is called the "Look-and-say sequence":

1, 11, 21, 1211, 111221, 312211, ...

- What is the rule for this pattern?
- Will the number 4 ever appear as a digit in this sequence?