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Centre for Education in
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Grade 11/12 Math Circles
Problem Set
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1. Imagine an infinite queue of guests waiting to check in to the Hilbert Hotel (a hotel with infinitely many rooms, numbered 1, 2, 3, ..., i.e. a room for each natural number).

   At 11 p.m. the receptionist admits two guests to the hotel and sends them to rooms 1 and 2. One of them hates her room and tells the receptionist she is going to another hotel. At 11:30 the receptionist again admits two guests to the hotel and sends them to vacant rooms, and one of the guests that are already in a room, decides to leave after this. This repeats over and over again at every time $(12 - \frac{n}{2})$ hour (e.g. {11:30, 11:45, 11:52:30, etc...} for every natural number n)

   How many rooms will be occupied at midnight under these conditions:

   (a) Every time a guest leaves their room, it is the guest who lives in the room with the lowest assigned number at that point in time.

   (b) The receptionist never reassigns a room that has been previously occupied.

   (c) For any new guests, the receptionist assigns them to **unoccupied** rooms with the lowest possible room number.

2. Prove that all the definitions below are equivalent (i.e. prove that for any two of them, one implies the other):

   (a) X is an infinite set. That is, there exists $Y \subset X$ $(Y \neq X)$ such that $X \sim Y$

   (b) There exists a function $f : X \to X$ that is injective but not surjective.

   (c) There exists a function $f : X \to X$ that is surjective but not injective.

   (d) For every natural number n, there is an injective function f from $\{1, 2, 3, \ldots, n\}$ to X.

   (e) For every natural number n, there is an injective function f from $\mathbb{N}$ to X.

   (f) For any $A \subseteq \mathbb{N}$, $X \sim (X \cup A)$

3. (HARD) Find a bijection from $[0, 1]$ to $(0, 1)$.

Recall that a set is *countable* if it is equivalent to a subset of the natural numbers. Otherwise, it is *uncountable*.

4. Determine which of the following sets are countable and which are uncountable.

   (a) The set of all possible strings (a string refers to any combination of the letters of the lowercase alphabet).

   (b) $\mathbb{Q} \times \mathbb{Q}$

   (c) The set of all lines which pass through the origin (i.e. all lines $y = mx$).

   (d) The set of all lines in the plane that pass through at most one rational point.
   (A point (x, y) is rational if x, y are both rationals).

   (e) The set of all lines in the plane that do not pass through any rational point.

   (f) The set of all increasing sequences of natural numbers (sequences $\{a_1, a_2, a_3, \ldots, a_n\}$ such that $a_{n+1} > a_n$ for all natural numbers n)

   (g) The set of all increasing sequences of natural numbers (sequences $\{a_1, a_2, a_3, \ldots, a_n\}$ such that $a_{n+1} < a_n$ for all natural numbers n)

   (h) (PRETTY HARD) The set of all *weakly* decreasing sequences of natural numbers (sequences $\{a_1, a_2, a_3, \ldots, a_n\}$ such that $a_{n+1} \leq a_n$ for all natural numbers n)

5. A collection of open intervals is said to be *disjoint* if they do not overlap - i.e. for any two intervals $(a, b), (c, d)$ we have $(a, b) \cap (c, d) = \emptyset$.

   Prove that any collection of open intervals is countable.

6. (REQUIRES KNOWLEDGE OF CALCULUS) Prove that if f is a monotonically increasing function (i.e. if $x < y$ then $f(x) < f(y)$) then the set $\{x \mid f(x) \text{ is not continuous at } x\}$ is countable.

7. (HARD) Prove that there exists a point (x, y) in the plane such that for any radius r, there exists at most one rational point in the plane whose distance to (x, y) is exactly r.

8. (VERY HARD) A set of sets, C, is called a *chain* if for any sets $x, y \in C$, either $x \subseteq y$ or $y \subseteq x$. Prove there exists a chain C of sets of natural numbers that is uncountable.

9. (EXTREMELY HARD) A set of sets, A is called an *anti-chain* if for any sets $x, y \in C$, neither is a subset of the other. Prove there exists a chain C of sets of natural numbers that is uncountable.