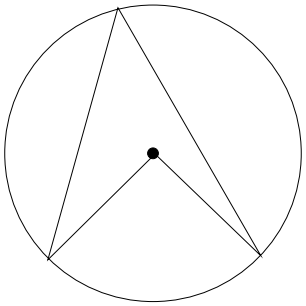
UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

50 YRS 1963-2013

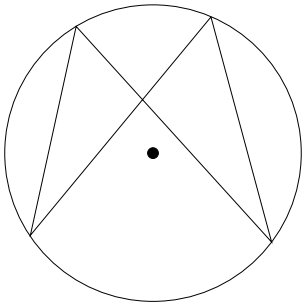
Enriching Mathematics and Computer Science for 50 years

Senior Math Circles
Winter 2013
Inversive

Theorem. An angle at the circumference of a circle equals one-half the angle at the center subtended by the same arc.

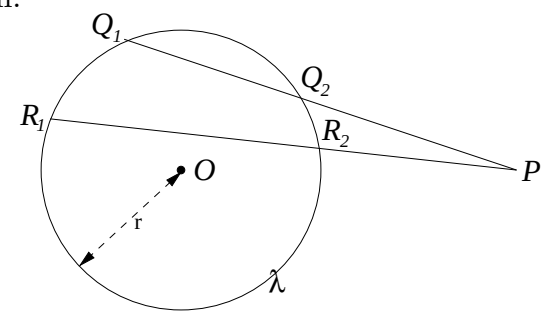


Theorem. Angles on the same arc (or equal arcs) of a circle are equal.



Theorem.

Given:



Then $PQ_1 \cdot PQ_2 \;=\; PR_1 \cdot PR_2 \;=\; PO^2 - r^2$.

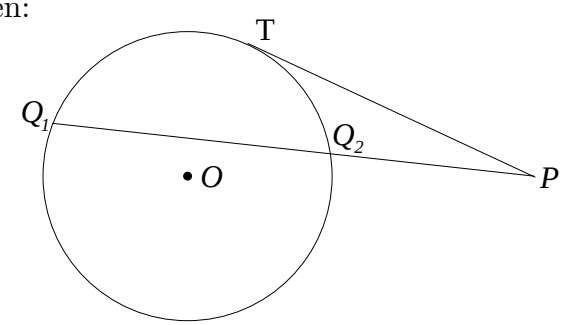
Definition: This number is called the **power** of P with respect to the circle.

If the circle is λ, write: $\mathcal{P}(P, \lambda)$

Proof follows from ....

Theorem.

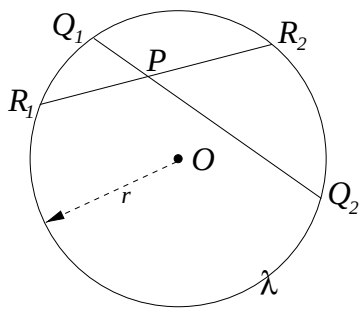
Given:



Then $PQ_1 \cdot PQ_2 \;= PT^2 \;= \mathcal{P}(P, \lambda)$.

<u>Proof.</u>

If P is inside the circle:



Then $-PQ_1 \cdot PQ_2 \;=\; -PR_1 \cdot PR_2 \;=\; PO^2 - r^2$.

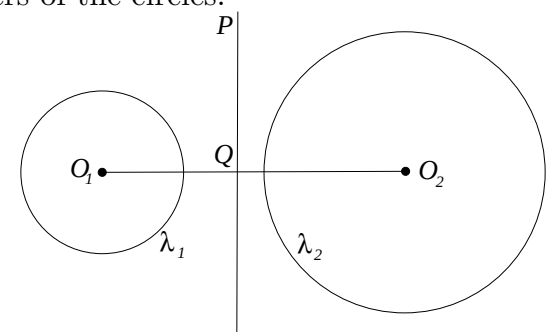
This number is again called the power of P with respect to the circle.

The power of P with respect to the circle λ, $\mathcal{P}(P, \lambda)$

- is positive if P is outside λ
- is negative if P is inside λ
- if P is on λ?

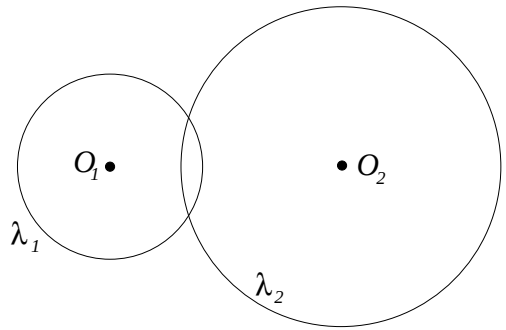
Definition. The **radical axis** of two circles is the locus of all points P whose power is the same with respect to the two circles.

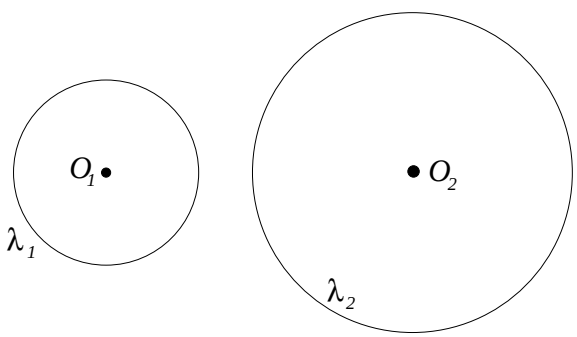
Theorem. The radical axis of two circles is a straight line perpendicular to the line through the centers of the circles.



<u>Proof.</u>

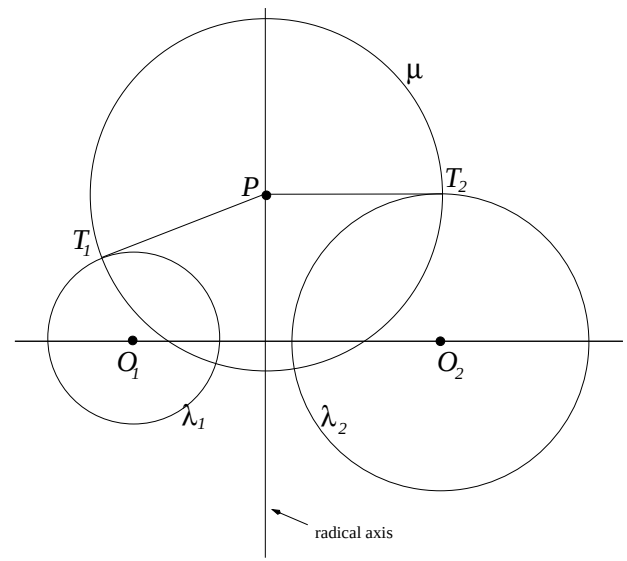
Constructing the radical axis.

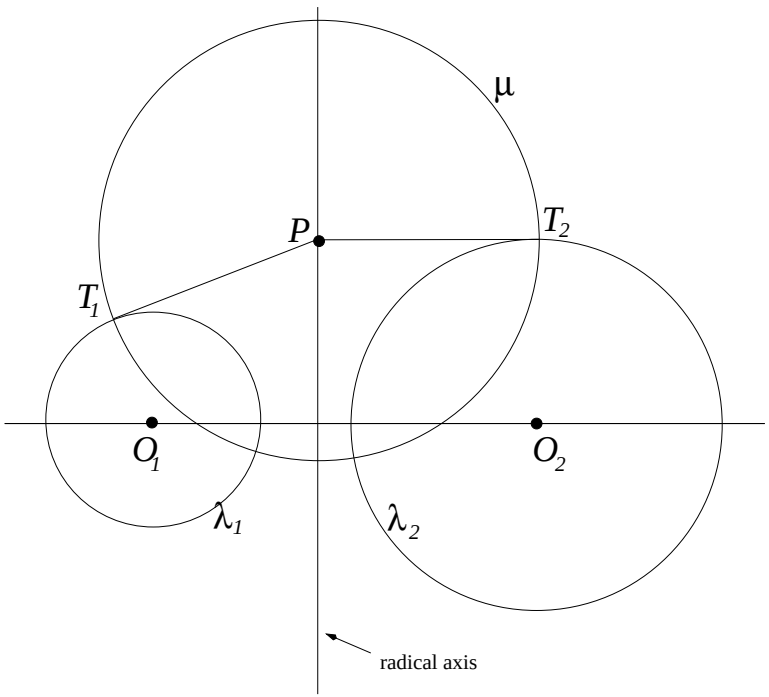


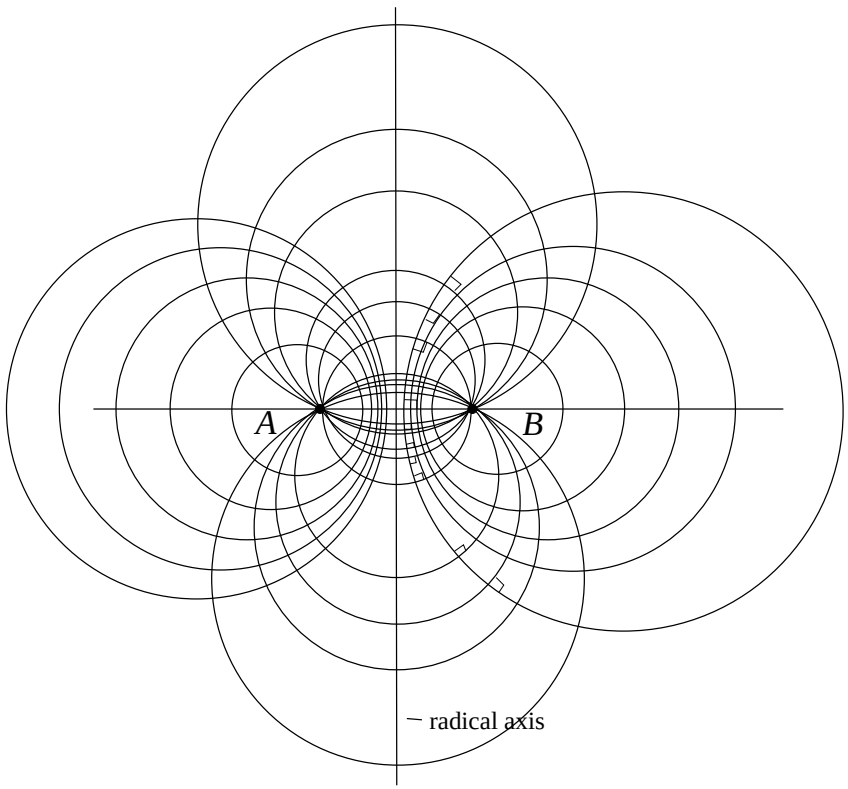


Big deal about the radical axis:

There is a circle μ centered at P which cuts circles λ_1, λ_2 at right angles if and only if P is on the radical axis of circles λ_1, λ_2.
Also say μ is **orthogonal** to λ_1, λ_2.

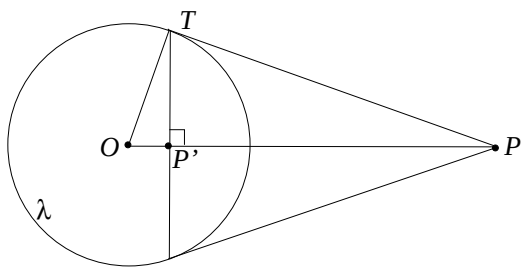



μ
P
T2
T1
O1
O2
λ1
λ2
radical axis


A
B
radical axis

Theorem.

Given



Then $PO \cdot P'O = r^2$

<u>Proof</u>

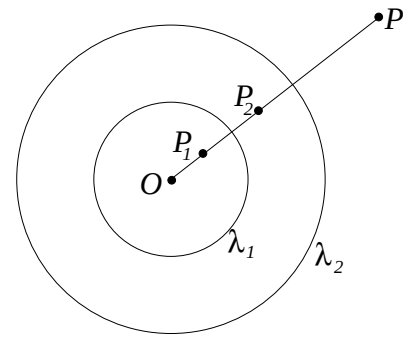
Points P and P' above are said to be **inverses** of each other, with respect to the circle λ.

Write P' for the inverse of P if the circle is clear. Otherwise write $I_\lambda(P)$.

$P'' = (P')' = P$

$O' =??$

Theorem. Let λ_1, λ_2 be concentric circles, with radii r_1, r_2. Then inversion in λ_1 and inversion in λ_2 differ by a similarity: either a contraction or dilation by the factor of $\frac{{r_2}^2}{{r_1}^2}$

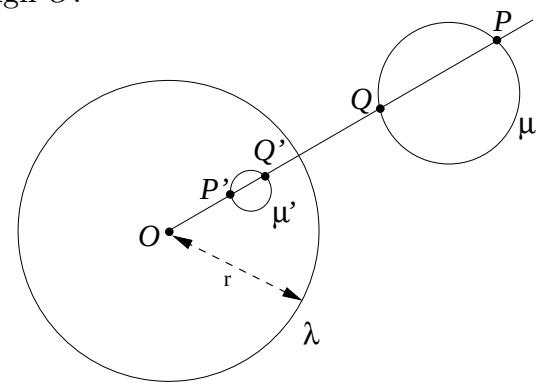


P_1 inverse of P in λ_1
P_2 inverse of P in λ_2

Then $OP_2 = \frac{{r_2}^2}{{r_1}^2} \cdot OP_1$

<u>Proof</u>

Theorem. The inverse of a circle μ *not* passing through the center of inversion O is a circle. The line through the centers of μ and μ' passes through O.



<u>Proof</u>

Theorem. The inverse of a line l *not* through the center of inversion is a circle passing through the center of inversion.

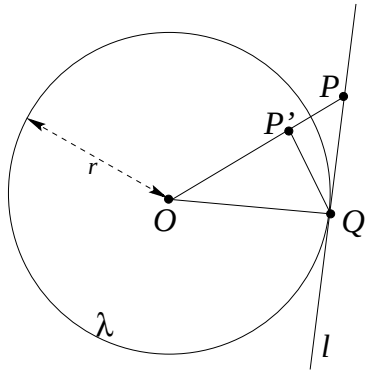
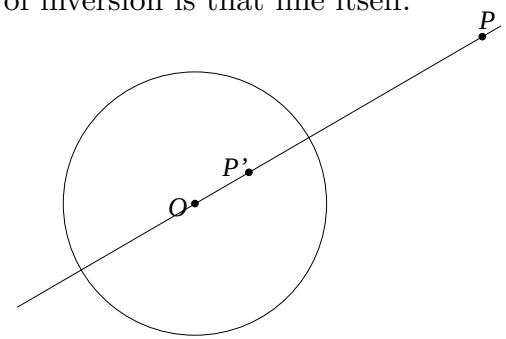


Diagram is a special case!

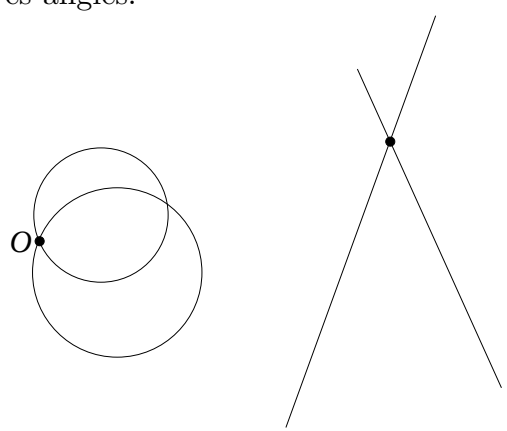
<u>Proof</u>

Conversely, the inverse of a circle passing through the center of inversion is a line not passing through the center of inversion.

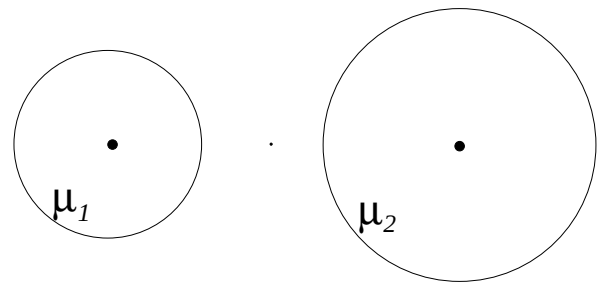
Theorem. The inverse of a line through the center of inversion is that line itself.



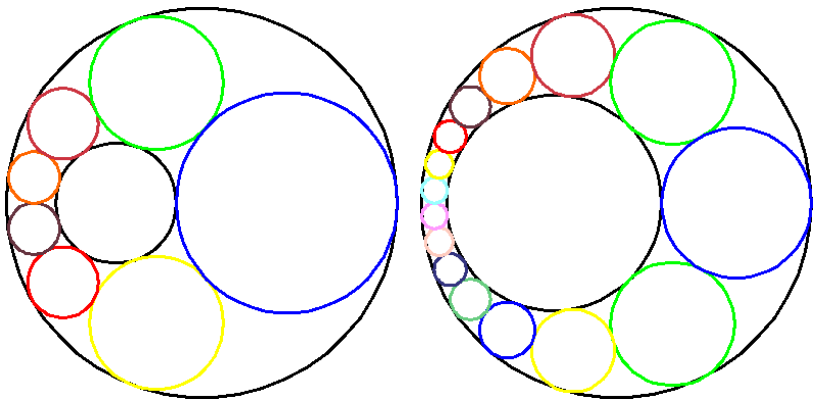
Theorem. Inversion is *conformal*. That is, preserves angles.



Theorem. Suppose μ_1 and μ_2 are non-overlapping circles. Then there is an inversion that transforms them into concentric circles.



Steiner chains of circles.



Chain of length 7 Chain of length 15

Steiners Porism . If two non overlapping circles admit one Steiner chain, they admit one of the same length starting with any circle that is between the first two circles, and tangent to both.

<u>Proof.</u>

Invert the two black circles into concentric circles.

The result is then obvious!