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Senior Math Circles

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Problem Solving, Part 2 - Solutions

1. Prove that $n^2 + 3n + 5$ is never divisible by 121 for any positive integer n.

   Solution: Suppose to the contrary that 121 divides $n^2 + 3n + 5$ for some positive integer n.

   Then, $n^2 + 3n + 5 = 121k$ for some positive integer k.

   Thus, $n^2 + 3n + (5 - 121k) = 0$. Using the quadratic formula, we have

$$\begin{aligned} n &= \frac{-3 \pm \sqrt{9 - 20 + 484k}}{2} \\ &= \frac{-3 \pm \sqrt{11(44k - 1)}}{2} \end{aligned}$$

   In order for n to be an integer, we must have $11 \mid 44k - 1$.

   However $44k - 1 \equiv 1 \not\equiv 0 \pmod{11}$. Thus, $11 \not\mid 44k - 1$, and we have reached a contradiction. Therefore, $n^2 + 3n + 5$ is never divisible by 121 for any positive integer n. ■

2. Prove that $n!+1$ and $(n+1)!+1$ are always relatively prime numbers, for any positive integer n.

   Solution: By way of contradiction, suppose there is a prime $p > 1$ such that $p \mid n! + 1$ and $p \mid (n + 1)! + 1$.

   Then

$$p \mid (n! + 1 - ((n + 1)! + 1))$$

   so

$$p \mid n(n!).$$

   But if $p \mid n$, then $p \mid n!$. Thus, $p \mid n!$.

   However, since $p \mid n! + 1$, this is a contradiction. Thus, no such prime p can exist, and thus, $n! + 1$ and $(n + 1)! + 1$ are always relatively prime numbers. ■

3. Prove that if $ax^2 + bx + c = 0$ has real solutions for x and that a, b, c are not zero, then a, b and c cannot form a geometric sequence.

   Solution: Suppose, by way of contradiction, that a, b, c do form an geometric sequence. In other words, $a, b = ar, c = ar^2$ is a sequence for some number r.

   We know that if $ax^2 + bx + c = 0$ has real solutions for x, then the discriminant $b^2 - 4ac \geq 0$.

   Since $b = ar$ and $c = ar^2$, we know that $(ar)^2 - 4a(ar^2) = a^2r^2 - 4a(ar^2) = -3a^2r^2 > 0$.

   However, $a^2 > 0$ and $r^2 > 0$, so $-3a^2r^2 < 0$, which is a contradiction. Thus, a, b, c cannot be consecutive terms in a geometric sequence. ■

4. What is the largest two-digit number that becomes 75% greater when its digits are reversed?

   Solution: [Euclid 2011, Q5(a)]

   Let n be the original number and N be the number when the digits are reversed. Since we are looking for the largest value of n, we assume that n ¿ 0.

   Since we want N to be 75% larger than n, then N should be 175% of n, or $N = \frac{7}{4}n$.

   Suppose that the tens digit of n is a and the units digit of n is b. Then $n = 10a + b$. Also, the tens digit of N is b and the units digit of N is a, so $N = 10b + a$. We want $10b + a = \frac{7}{4}(10a + b)$ or $4(10b + a) = 7(10a + b)$ or $40b + 4a = 70a + 7b$ or $33b = 66a$, and so $b = 2a$.

   This tells us that that any two-digit number $n = 10a + b$ with $b = 2a$ has the required property.

   Since both a and b are digits then $b < 10$ and so $a < 5$, which means that the possible values of n are 12, 24, 36, and 48. The largest of these numbers is 48. ■

5. Oi-Lam tosses three fair coins and removes all of the coins that come up heads. George then tosses the coins that remain, if any. Determine the probability that George tosses exactly one head.

   Solution: [Euclid 2010, 8(a)]

   If Oi-Lam tosses 3 heads, then George has no coins to toss, so cannot toss exactly 1 head.

   If Oi-Lam tosses 2, 1 or 0 heads, then George has at least one coin to toss, so can toss exactly 1 head.

   Therefore, the following possibilities exist:

   - Oi-Lam tosses 2 heads out of 3 coins and George tosses 1 head out of 1 coin
   - Oi-Lam tosses 1 head out of 3 coins and George tosses 1 head out of 2 coins
   - Oi-Lam tosses 0 heads out of 3 coins and George tosses 1 head out of 3 coins

   We calculate the various probabilities.

   If 3 coins are tossed, there are 8 equally likely possibilities: HHH, HHT, HTH, THH, TTH, THT, HTT, TTT. Each of these possibilities has probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Therefore,

   - the probability of tossing 0 heads out of 3 coins is $\frac{1}{8}$
   - the probability of tossing 1 head out of 3 coins is $\frac{3}{8}$
   - the probability of tossing 2 heads out of 3 coins is $\frac{3}{8}$
   - the probability of tossing 3 heads out of 3 coins is $\frac{1}{8}$

   If 2 coins are tossed, there are 4 equally likely possibilities: HH, HT, TH, TT. Each of these possibilities has probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore, the probability of tossing 1 head out of 2 coins is $\frac{2}{4} = \frac{1}{2}$.

   If 1 coin is tossed, the probability of tossing 1 head is $\frac{1}{2}$.

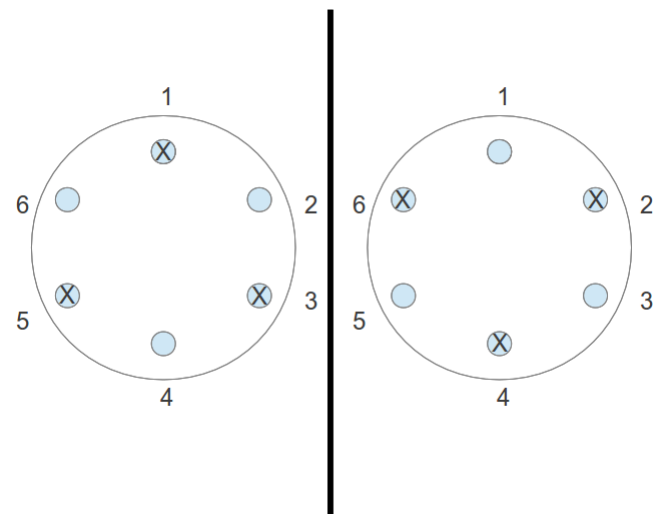
   To summarize, the possibilities are

   - Oi-Lam tosses 2 heads out of 3 coins (with probability $\frac{3}{8}$) and George tosses 1 head out of 1 coin (with probability $\frac{1}{2}$)

- Oi-Lam tosses 1 head out of 3 coins (with probability $\frac{3}{8}$) and George tosses 1 head out of 2 coins (with probability $\frac{1}{2}$)
- Oi-Lam tosses 0 heads out of 3 coins (with probability $\frac{1}{8}$) and George tosses 1 head out of 3 coins (with probability $\frac{3}{8}$)

Therefore, the overall probability is $\frac{3}{8} \cdot \frac{1}{2} + \frac{3}{8} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{3}{8} = \frac{27}{64}$. ■

6. Suppose there are n plates equally spaced around a circular table. Ross wishes to place an identical gift on each of k plates, so that no two neighbouring plates have gifts. Let $f(n, k)$ represent the number of ways in which he can place the gifts. For example $f(6, 3) = 2$, as shown below:



(a) Determine the value of $f(7, 3)$.

(b) Prove that $f(n, k) = f(n - 1, k) + f(n - 2, k - 1)$ for all integers $n \geq 3$ and $k \geq 2$.

Solution: [Euclid 2009, 10(a)+(b)]

Throughout this problem, we represent the states of the n plates as a string of 0s and 1s (called a *binary string*) of length n of the form $p_1 p_2 \cdots p_n$, with the rth digit from the left (namely p_r) equal to 1 if plate r contains a gift and equal to 0 if plate r does not. We call a binary string of length n *allowable* if it satisfies the requirements – that is, if no two adjacent digits both equal 1. Note that digit p_n is also "adjacent" to digit p_1, so we cannot have $p_1 = p_n = 1$.

(a) Suppose that $p_1 = 1$. Then $p_2 = p_7 = 0$, so the string is of the form $10p_3p_4p_5p_60$. Since $k = 3$, then 2 of p_3, p_4, p_5, p_6 equal 1, but in such a way that no two adjacent digits are both 1. The possible strings in this case are 1010100, 1010010 and 1001010.

Suppose that $p_1 = 0$. Then p_2 can equal 1 or 0. If $p_2 = 1$, then $p_3 = 0$ as well. This means that the string is of the form $010p_4p_5p_6p_7$, which is the same as the general string in the first case, but shifted by 1 position around the circle, so there are again 3 possibilities. If $p_2 = 0$, then the string is of the form $00p_3p_4p_5p_6p_7$ and 3 of the digits p_3, p_4, p_5, p_6, p_7 equal 1 in such a way that no 2 adjacent digits equal 1. There is only 1 way in which this can happen: 0010101. Overall, this gives 7 possible configurations, so $f(7, 3) = 7$.

(b) An allowable string $p_1p_2\cdots p_{n-1}p_n$ has $(p_1, p_n) = (1, 0), (0, 1)$, or $(0, 0)$.

Define $g(n, k, 1, 0)$ to be the number of allowable strings of length n, containing k 1s, and with $(p_1, p_n) = (1, 0)$.

We define $g(n, k, 0, 1)$ and $g(n, k, 0, 0)$ in a similar manner.

Note that $f(n, k) = g(n, k, 1, 0) + g(n, k, 0, 1) + g(n, k, 0, 0)$.

Consider the strings counted by $g(n, k, 0, 1)$.

Since $p_n = 1$, then $p_{n-1} = 0$. Since $p_1 = 0$, then p_2 can equal 0 or 1.

We remove the first and last digits of these strings.

We obtain strings $p_2p_3\cdots p_{n-2}p_{n-1}$; that is strings of length $n-2$ containing $k-1$ 1s.

Since $p_{n-1} = 0$, then the first and last digits of these strings are not both 1. Also, since the original strings did not contain two consecutive 1s, then these new strings do not either.

Therefore, $p_2p_3\cdots p_{n-2}p_{n-1}$ are allowable strings of length $n-2$ containing $k-1$ 1s, with $p_{n-1} = 0$ and $p_2 = 1$ or $p_2 = 0$.

The number of such strings with $p_2 = 1$ and $p_{n-1} = 0$ is $g(n-2, k-1, 1, 0)$ and the number of such strings with $p_2 = 0$ and $p_{n-1} = 0$ is $g(n-2, k-1, 0, 0)$.

Thus, $g(n, k, 0, 1) = g(n-2, k-1, 1, 0) + g(n-2, k-1, 0, 0)$.

Consider the strings counted by $g(n, k, 0, 0)$. Since $p_1 = 0$ and $p_n = 0$, then we can remove p_n to obtain strings $p_1p_2\cdots p_{n-1}$ of length $n-1$ containing k 1s. These strings are allowable since $p_1 = 0$ and the original strings were allowable.

Note that we have $p_1 = 0$ and p_{n-1} is either 0 or 1.

So the strings $p_1p_2\cdots p_{n-1}$ are allowable strings of length $n-1$ containing k 1s, starting with 0, and ending with 0 or 1.

The number of such strings with $p_1 = 0$ and $p_{n-1} = 0$ is $g(n-1, k, 0, 0)$ and the number of such strings with $p_1 = 0$ and $p_{n-1} = 1$ is $g(n-1, k, 0, 1)$.

Thus, $g(n, k, 0, 0) = g(n-1, k, 0, 0) + g(n-1, k, 0, 1)$.

Consider the strings counted by $g(n, k, 1, 0)$.

Here, $p_1 = 1$ and $p_n = 0$. Thus, p_{n-1} can equal 0 or 1. We consider these two sets separately.

If $p_{n-1} = 0$, then the string $p_1p_2\cdots p_{n-1}$ is an allowable string of length $n-1$, containing k 1s, beginning with 1 and ending with 0.

Therefore, the number of strings counted by $g(n, k, 1, 0)$ with $p_{n-1} = 0$ is equal to $g(n-1, k, 1, 0)$.

If $p_{n-1} = 1$, then the string $p_2p_3\cdots p_{n-1}$ is of length $n-2$, begins with 0 and ends with 1.

Also, it contains $k-1$ 1s (having removed the original leading 1) and is allowable since the original string was.

Therefore, the number of strings counted by $g(n, k, 1, 0)$ with $p_{n-1} = 1$ is equal to $g(n-2, k-1, 0, 1)$.

Therefore,

$$\begin{aligned}
f(n,k) &= g(n,k,1,0) + g(n,k,0,1) + g(n,k,0,0) \\
&= (g(n-1,k,1,0) + g(n-2,k-1,0,1)) \\
&\quad +(g(n-2,k-1,1,0) + g(n-2,k-1,0,0)) \\
&\quad +(g(n-1,k,0,0) + g(n-1,k,0,1)) \\
&= (g(n-1,k,1,0) + g(n-1,k,0,1) + g(n-1,k,0,0)) \\
&\quad +(g(n-2,k-1,0,1) + g(n-2,k-1,1,0) + g(n-2,k-1,0,0)) \\
&= f(n-1,k) + f(n-2,k-1).
\end{aligned}$$

as required.