Grade 7/8 Math Circles
February 4/5, 2014
Sets - Solutions

Exercises I

1. Are the following statements true or false? If false, explain why:

   (a) Is $\{1, 1, 2, 3, 4, 5\}$ a valid set based on what we learned in this lesson?

   FALSE: there is no uniqueness due to a repeated 1, so it is not a valid set

   (b) Is the set $\{1, 3, 5, 7, 9\}$ the same as the set $\{1, 3, 5, 9, 7\}$?

   TRUE

   (c) Is the set $\{3, 6, 9, 12, 15\}$ the same as the set $\{3, 6, 9, 12, 15...\}$?

   FALSE: The second set's "..." tells us that the set continues the pattern of multiples of three.

2. Fill in the blanks with a $\in, \ni, \subseteq, \supseteq$ or $=$ to make the sentence true:

   (a) Dark Red $\in$ { Shades of Red }

   (b) { Shades of Blue } $\subseteq$ { Colours } $\ni$ Green

   (c) Dog $\in$ { Pets } $\ni$ Cat

   (d) { 100, 99, 98 ... 2, 1 } $=$ { 1, 2, ... 98, 99, 100 } $\supseteq$ { 2, 4, ... 98, 100 }

   (e) { Positive Even Numbers } $\subseteq$ { ...-2, 0, 2, 4, 6, 8 ... } $=$ { Multiples of 2 }

Exercises II

1. Let A = { 1, 3, 5, 7, 9 }, B = { 2, 4, 6, 8, 10 } and C = { 2, 3, 5, 7, 11 }. Write out the following sets if $\mathbb{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$:

   (a) $A \cup C$

   $A \cup C = \{1, 2, 3, 5, 7, 9, 11\}$

(b) $B \cap C$

$B \cap C = \{2\}$

(c) $(A \cap C) \cup B$

$(A \cap C) \cup B = \{2, 3, 4, 5, 6, 7, 8, 10\}$

(d) $\overline{A}$

$\overline{A} = \{2, 4, 6, 8, 10, 11\}$

2. Let $R = \{$ Numbers Between 2 and 5, and Numbers Greater than 9$\}$. Find $\overline{R}$.

$\overline{R} = \{$All Numbers less than or equal to 2, and all Numbers greater than or equal to 5 and less than or equal to 9$\}$

Problems Sets

1. Give 5 elements of the following sets, or explain why you can't. Are there any pairs of subsets and supersets?:

   (a) $\{$Multiples of 3$\} \cap \{$Multiples of 4$\}$

   Multiple Answers Ex: $\{12, 24, 36, 48, 60\}$

   (b) $\{$Integers greater than 4$\} \cap \{$Integers less than 6$\}$

   There are not enough integers between 4 and 6 to complete the question

   (c) $\{$Factors of 100$\} \cap \{$Factors of 20$\}$

   Multiple Answers Ex: $\{1, 2, 4, 5, 10\}$, $\{$Factors of 20$\} \subseteq \{$Factors of 100$\}$

   (d) $\{$Even Numbers$\} \cap \{$Prime Numbers$\}$

   2 is the only example of an even, prime number - therefore not enough numbers to answer

   (e) $\{$Letters in the word "Because"$\} \cap$
   $\{$Letters in the poem "The Quick Brown Fox Jumps Over the Lazy Dog$\}$

   Multiple Answers Ex: $\{b, e, c, a, u\}$, $\{$Letters in the word "Because"$\} \subseteq$
   $\{$Letters in the poem "The Quick Brown Fox Jumps Over the Lazy Dog$\}$

2. Let $\mathbb{U}$ be the Universal Set and A be a set within the Universal Set, in words:

   (a) What is $\overline{\overline{\mathbb{U}}}$?

(b) What is $A \cap \overline{A}$?

(c) What is $A \cup \overline{A}$?

(d) What is $\overline{\overline{A}}$?

(e) What is $\{\ \} \cap \{\ \}$?

3. Like we did in the *Complements of Unions and Intersections* section, find the complement of $A \cup B$ step-by-step:

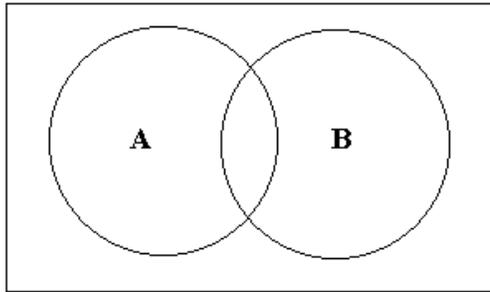Let's first put a bar over the entire statement to give us: $\overline{A \cup B}$
If we split up the bar we see: $\overline{A} \ \overline{\cup} \ \overline{B}$
From above we know that $\overline{\cup}$ becomes $\cap$ so we get: $\overline{A} \cap \overline{B}$

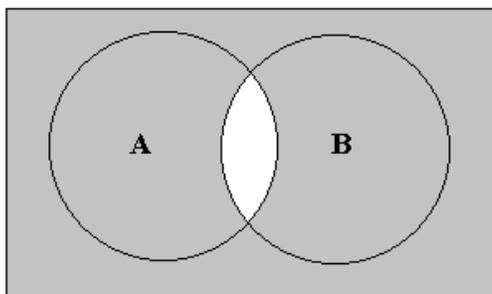We now know that the complement of $A \cup B$ is the intersection of $\overline{A}$ and $\overline{B}$!

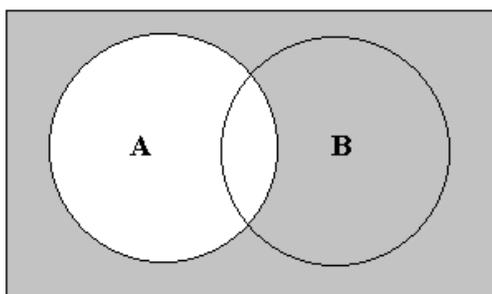4. (a) Draw a Venn Diagram where A intersects B:

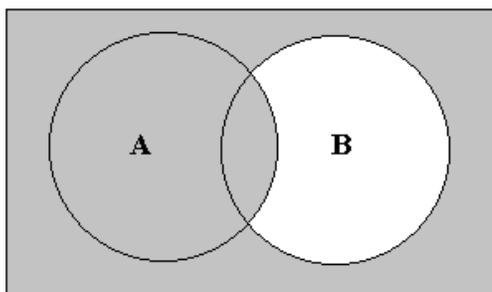
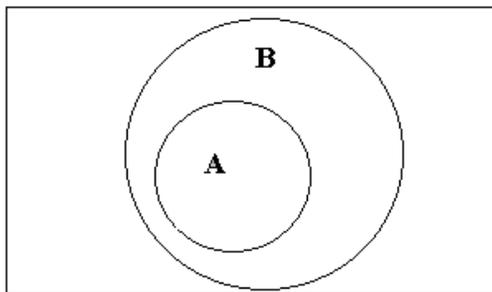(b) Shade in $\overline{A \cup B}$.
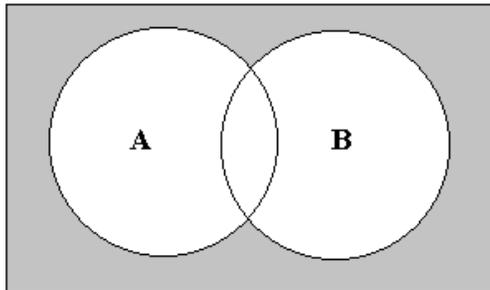
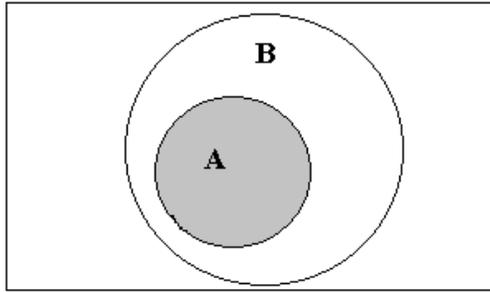(c) Shade in $\overline{A \cap B}$.

(d) Shade in $\overline{A}$.
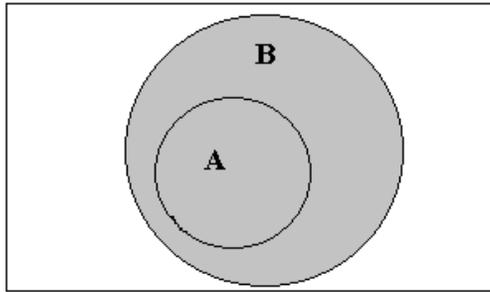
(e) Shade in $A \cup \overline{B}$.

5. (a) Draw a Venn Diagram where A is a subset of B:

(b) Shade in $A \cap B$.

(c) Shade in $A \cup B$.

(d) Shade in $\overline{A \cap B}$.
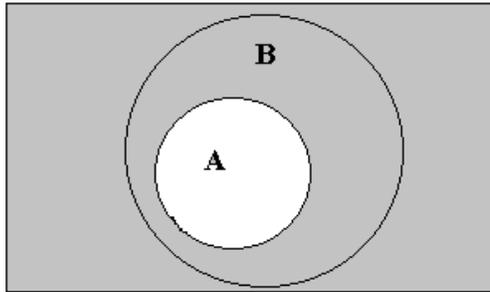
(e) Shade in $A \cap \overline{B}$.

Not possible - no answer

6. In a group of 60 people, 27 like cold drinks and 42 like hot drinks and each person likes at least one of the two drinks. How many like both hot drinks AND cold drinks?

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$60 = 27 + 42 - x$$
$$x = 9$$

7. Each student in a class of 40 plays at least one card game: Euchre, GoFish and Crazy8s. 18 play Euchre, 20 play GoFish and 27 play Crazy8s. 7 play Euchre and GoFish, 12 play GoFish and Crazy8s and 4 play Euchre, Crazy8s and GoFish.

(a) Find the number of students who play Euchre and Crazy8s. (Hint: Use the Inclusion-Exclusion Principle for three sets)

Let A = Euchre, B = GoFish and C = Crazy8s

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$40 = 18 + 20 + 27 - 7 - 12 - |A \cap C| + 4$$
$$|A \cap C| = 10$$

(b) How many students play only Euchre?
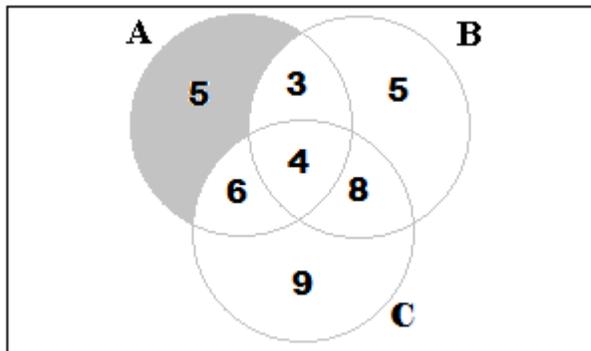
8. * Is the size of the set {Even Numbers} equal to the size of the set {Odd Numbers}?

Yes - *Multiple Answers*

9. Let $A = \{E, H, J, T, S, G, Z, N\}$, $B = \{L, Y, U, I, O, Z, N, R, K\}$, $C = \{G, Q, N, F, R, K, Y, F, Z, S, H, J, T\}$ and $D = \{U, I, N, G, F, X, R, Z, K, V\}$:

(a) Redraw the 4-set Venn Diagram shown to the right, and place all the elements in their proper section.

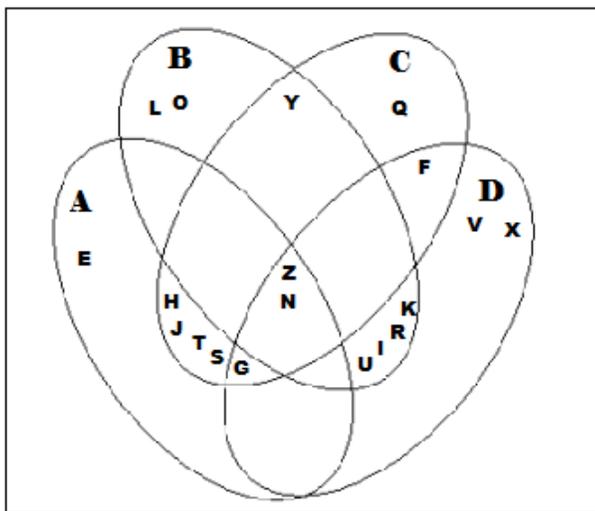(b) How many of the sections in the diagram to the right represent the intersection of 2 sets? 3 sets? All 4 sets?

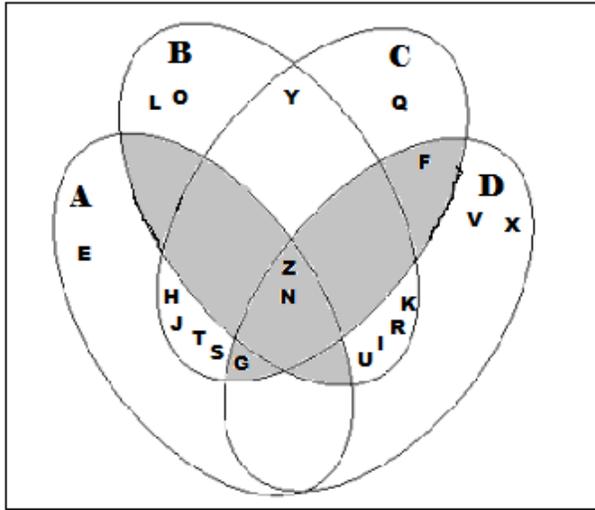2 sets: 6 representations)
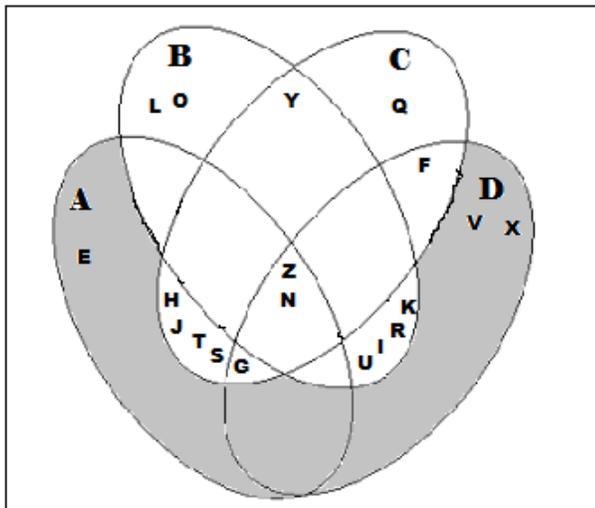
3 sets: 4 representations)

1 set: 1 representation

(c) What elements are in the set $(A \cap B) \cup (C \cap D)$? Shade this in on a Venn Diagram.

$(A \cap B) \cup (C \cap D) = \{F\}$

(d) What elements are in the set $(\overline{B} \cap \overline{C}) \cap (A \cup D)$? Shade this in on a Venn Diagram.

$(\overline{B} \cap \overline{C}) \cap (A \cup D) = \{F, Z, C, N\}$

10. In a restaurant, 40 people order fries. 13 people put salt on their fries, 28 people put ketchup on their fries, and 6 people do both. How many people eat their fries plain?

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$40 = 13 + 28 - 6$$
$$40 - 35 = 5$$

11. * Write the Inclusion-Exclusion Principle for $|A \cup B \cup C \cup D|$.

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| - |A \cap B \cap C \cap D|$

12. * Construct a Venn Diagram: 200 volleyball players were asked which of these moves they considered their weakest move(s): the serve, the bump, the spike.

- 20 players said none of these were their weakest moves.
- 30 players said all three of these were their weakest moves.
- 40 players said their serve and spike were their weakest moves.
- 40 players said that only their serve and bump were their weakest moves.
- 15 players said that their bump but not their spike was their weakest move.
- 52 players said that only their spike was their weakest move.
- 115 players said their serve was their weakest move.
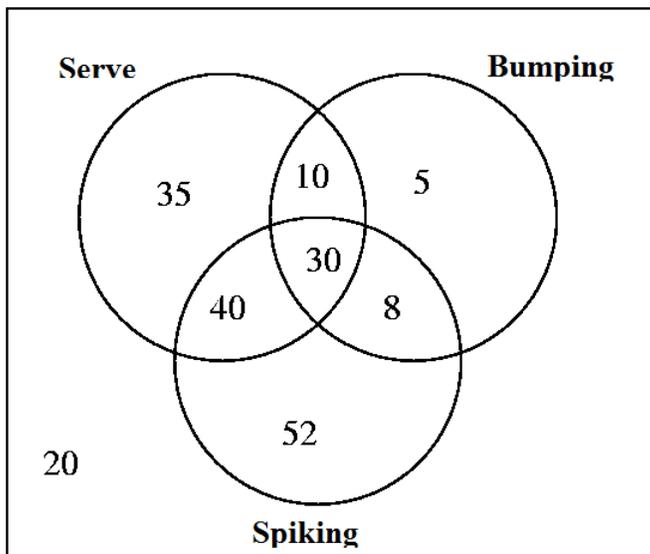
13. * * Construct a Venn Diagram: 150 people at a Justin Bieber concert were asked if they knew how to play piano, drums or guitar.

- 18 people could play none of these instruments.

- 10 people could play all three of these instruments.

- 77 people could play drums or guitar but could not play piano.

- 73 people could play guitar.

- 49 people could play at least two of these instruments.

- 13 people could play piano and guitar but could not play drums.

- 21 people could play piano and drums.

14. * * Is the size of the set {Numbers from 0 to 1} equal to the size of the set {$-\infty$ to ∞}? (Hint: Remember that there are decimal numbers between 0 and 1)

Yes - *Multiple Answers*

15. * * Write the Inclusion=Exclusion Principle for $|A \cup B \cup C \cup D \cup E|$.

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| + |E| - |A \cap B| - |A \cap C| - |A \cap D| - |A \cap E| - |B \cap C| - |B \cap D| - |B \cap E| - |C \cap D| - |C \cap E| - |D \cap E| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap B \cap E| + |A \cap C \cap D| + |A \ca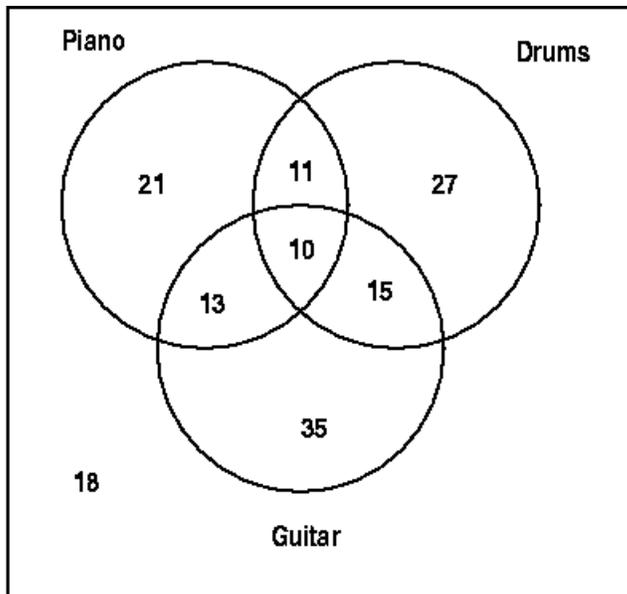p C \cap E| + |B \cap C \cap D| + |B \cap C \cap E| - |A \cap B \cap C \cap D| - |A \cap B \cap C \cap E| - |A \cap B \cap D \cap E| - |A \cap C \cap D \cap E| - |B \cap C \cap D \cap E| + |A \cap B \cap C \cap D \cap E|$

Grade 7/8 Math Circles
Sets and Venn Diagrams
Ｆｅｂｒｕａｒｙ 6, 2014

Problem Set

Part A - Concepts and Basics

1. (a) Is $\{1, 2, 3, 1, 5, 9, 10\}$ a valid set based on what we've learned? Why or why not?

   FALSE: there is no uniqueness due to a repeated 1, so it is not a valid set

   (b) Is the set $\{1, 3, 5, 7, 9\}$ the same as the set $\{1, 3, 5, 9, 7\}$?

   TRUE

   (c) Is the set $\{3, 6, 9, 12, 15\}$ the same as the set $\{3, 6, 9, 12, 15, ...\}$?

   FALSE: The second set's "..." tells us that the set continues the pattern of multiples of three.

2. Is $6 \in \{\text{the set of all even numbers}\}$?

   YES - since 6 is even.

3. Is the set $\{a, b, f, d\}$ contained in the set $\{a, b, c, f\}$?

   NO - containment means all or nothing.

4. * If our universe $\mathbb{U}$ is the set of all positive whole numbers $\{1, 2, 3, 4, 5, \dots\}$

   (a) Is $\{2, 4, 6, 8, 10\} \subseteq \{1, 2, 3, 4, 5, \dots\}$?

   YES

   (b) Is the set of all odd positive numbers a subset of our universe?

   YES

5. What is $\{\} \cup \{\}$? Explain why in English.

   It is the empty set. When you combine two sets with nothing, you will get a set that has nothing.

6. * What is $\{\} \cap \{\}$? Explain why in English.

It is the empty set. The intersection is the set containing elements that are common to both sets. Nothing is common to both sets. So the intersection must be the empty set. (This also follows from our definition that any set intersect the empty set is the same set).

7. Give 5 elements of the following sets, or explain why you can't. Are there any pairs of subsets and supersets?:

   (a) {Multiples of 3} ∩ {Multiples of 4}

   Multiple Answers Ex: {12, 24, 36, 48, 60}

   (b) {Integers greater than 4} ∩ {Integers less than 6}

   There are not enough integers between 4 and 6 to complete the question

   (c) {Factors of 100} ∩ {Factors of 20}

   Multiple Answers Ex: {1, 2, 4, 5, 10}, {Factors of 20} ⊆ {Factors of 100}

   (d) {Even Numbers} ∩ {Prime Numbers}

   2 is the only example of an even, prime number - therefore not enough numbers to answer

   (e) {Letters in the word "Because"}∩ {Letters in the poem "The Quick Brown Fox Jumps Over the Lazy Dog}

   Multiple Answers Ex: {b, e, c, a, u}, {Letters in the word "Because"} ⊆ {Letters in the poem "The Quick Brown Fox Jumps Over the Lazy Dog}

8. * Let $R = $ {All Real Numbers (Between 2 and 5) or (Numbers Greater than 9)} $S = $ {All Real Numbers (between 3 and 6) or (Greater than 8)}. Give a description for

   (a) R^c and S^c

   $R^c = $ {All Real Numbers (Less than or Equal to 2) and (Between 5 and 9 Inclusive)}

   $S^c = $ {All Real Numbers (Less than or Equal to 3) and (Between 6 and 8 Inclusive)}

   (b) $R \cup S$ and $R \cap S$

   $R \cup S = $ {All Real Numbers (Between 2 and 6) or (Numbers Greater than 8)}

   $R \cap S = $ {All Real Numbers (Between 3 and 5) or (Numbers Greater than or equal to 9)}

   (c) $R^c \cup S$ and $R \cap S^c$

   $R^c \cup S = $ {All Real Numbers (Less than or Equal to 2) and (Greater than 3)}

11

$R \cap S^c = \{$All numbers Greater than 2 but Less than or equal to 3$\}$

(d) $(R \cup S)^c$ and $(R \cap S)^c$.

$(R \cup S)^c = \{$All Real Numbers (Less than or equal to 2) and (Numbers between 6 and 8 Inclusi

$(R \cap S)^c = \{$All Real Numbers (Less than or equal to 3) and (Numbers between 5 and 9 Inclusi

Let R = {Euclid, Riemann, Newton, Hilbert}, S = {Galois, Gauss, Abel, Euclid}, and T = {Hilbert, Riemann, Godel, Wiles, Chern}.

9. Let $\mathbb{U} = (R \cup S) \cup T$. What is this set?

$\mathbb{U} = \{$Euclid, Riemann, Newton, Hilbert, Galois, Gauss, Abel, Godel, Wiles, Chern)$\}$

(a) Using this as our universe, what is R^c?

$R^c = \{$Galois, Gauss, Abel, Godel, Wiles, Chern)$\}$

(b) What is $(S \cap T)^c$? What is $S^c \cup T^c$?

$(S \cap T)^c = \{$Euclid, Riemann, Newton, Hilbert, Galois, Gauss, Abel, Godel, Wiles, Chern)$\}$

$S^c \cup T^c = \{$Euclid, Riemann, Newton, Hilbert, Galois, Gauss, Abel, Godel, Wiles, Chern)$\}$

10. What is the set $(R \cup S) \cap (S \cap T)$?

$\{\}$ - The empty set

11. Using {R, S, T, $\cap$, $\cup$}, fill in the blanks to complete the equation:

$$(R \quad \cap \quad T) \quad \cup \quad (S \quad \cup \quad R) = \{\text{Riemann, Euclid, Hilbert}\}$$

Part B - Applying Inclusion Exclusion Principle

1. 40 students were surveyed about their favourite singer.

   - 24 liked Justin Bieber.
   - 12 liked Miley Cyrus.
   - 7 liked both.

How many students like one or the other?

$$\#JB + \#MC - \#(JB \cap MC) = \#(JB \cup MC)$$
$$24 + 12 - 7 = \#(JB \cup MC)$$
$$\therefore 29 = \#(JB \cup MC)$$

2. Sherlock Holmes has just called you in to solve a difficult mystery. A group of thieves are getting ready to rob the bank, but he does not know exactly how many will do the job.

  - 17 of them are known safecrackers.

  - 28 of them are known getaway drivers.

  - There are 40 thieves who are safecrackers and/or drivers.

Holmes knows that the group that will rob the bank will be those thieves who are both safecrackers and getaway drivers, but he didn't pay attention in math class and can't figure it out. How many people will be in this group?

$$\#S + \#G - \#(S \cap G) = \#(S \cup G)$$
$$17 + 28 - \#(S \cap G) = 40$$
$$45 - \#(S \cap G) = 40$$
$$\therefore \#(S \cap G) = 5$$

3. * There has been a mishap at the Hogwarts Guidance Office. A clumsy clerk has mixed up all of the Sorting Hat Data for 21 students. She was able to salvage the following data:

  - 13 of them are enrolled in Potions.

  - 12 are enrolled in Transfiguration.

  - 16 are enrolled in Charms.

  - 8 of them are enrolled in Potions and Transfiguration.

  - 7 are enrolled in Charms and Transfiguration

  - 9 are enrolled in Charms and Potions

  - 4 are enrolled in all three classes.

Draw a Venn Diagram of the situation. You can assume your universe is the set of all Hogwarts students. Name each of your sets with a letter in your diagram.
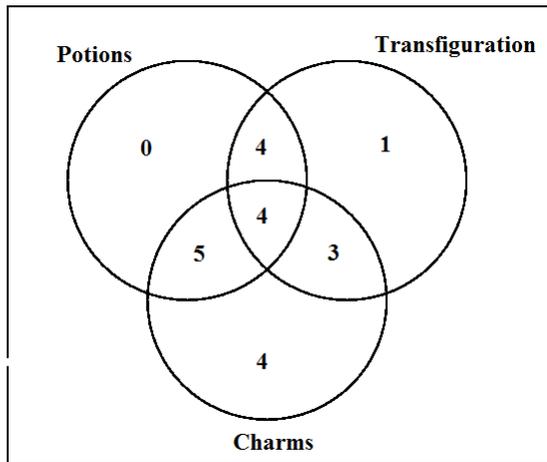
How many students are enrolled **only** in Potions?

No students are enrolled only in Potions.

4. In a restaurant, 40 people order fries. 13 people put salt on their fries, 28 people put ketchup on their fries, and 6 people do both. How many people eat their fries plain?

$$\#A + \#B - \#(A \cap B) = \#(A \cup B)$$
$$13 + 28 - 6 = 40$$
$$\therefore 5 = 40 - 35$$

5. ** 200 volleyball players were asked which of these moves they considered their weakest move(s): the serve, the bump, the spike.

- 20 players said none of these were their weakest moves.
- 30 players said all three of these were their weakest moves.
- 40 players said their serve and spike were their weakest moves.
- 40 players said that only their serve and bump were their weakest moves.
- 15 players said that their bump but not their spike was their weakest move.
- 52 players said that only their spike was their weakest move.
- 115 players said their serve was their weakest move.
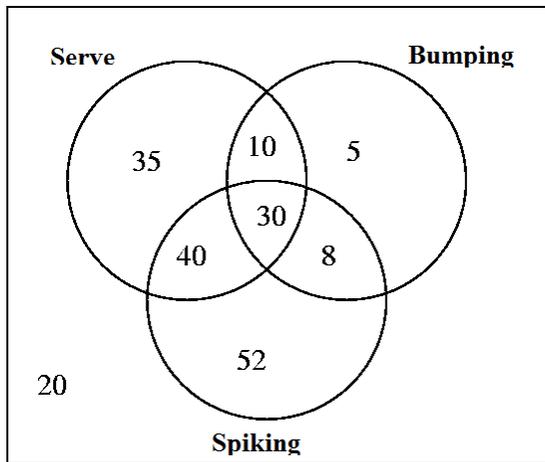
How many people listed bump, but not serve, as a weakest move?

13 people listed bump, but not serve, as a weakest move.

6. ** 150 people at a Justin Bieber concert were asked if they knew how to play piano, drums or guitar.

- 18 people could play none of these instruments.
- 10 people could play all three of these instruments.
- 77 people could play drums or guitar but could not play piano.
- 73 people could play guitar.
- 49 people could play at least two of these instruments.
- 13 people could play piano and guitar but could not play drums.
- 21 people could play piano and drums.
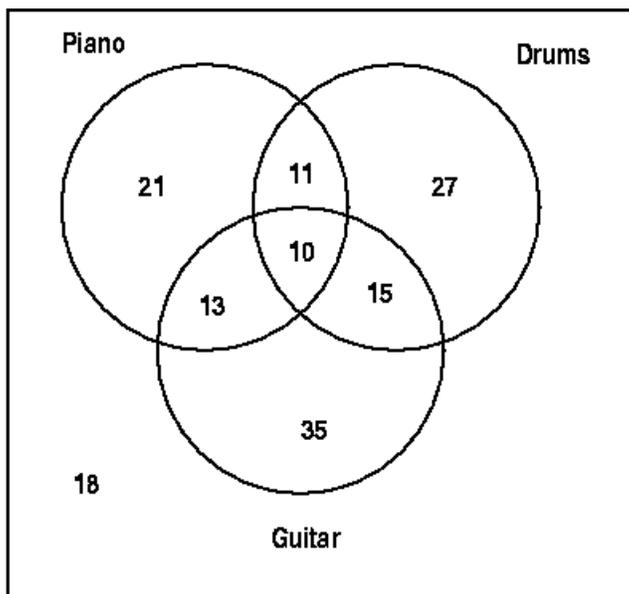
How many people play piano but do not play drums?

34 people play piano, but not the drums

Part C - Challenging Extensions

1. ** You know the formula for the Inclusion-Exclusion Principle of 2 sets... can you derive a similar formula for 3 sets? **Can you prove why this is true?**

The Inclusion Exclusion Principle for 3 sets P, T, C, is:
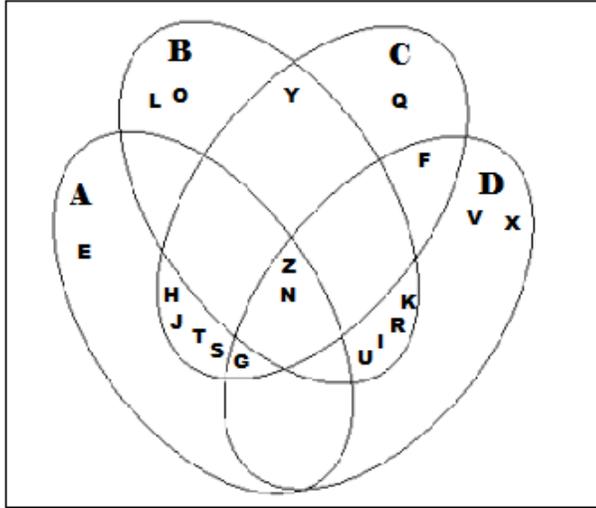
$$\#P + \#T + \#C - \#(P \cap T) - \#(P \cap C) - \#(T \cap C) + \#(P \cap T \cap C) = \#(P \cup T \cup C)$$

The first three terms happen when we add up the sizes of the three sets. (This is $\#P$ + $\#T$ + $\#C$). When we do this, though, we have to remember that any pair of these sets overlap with each other.

So we have to subtract all the times they overlap once, to eliminate double counting. This is why we subtract $\#(P \cap T)$, $\#(P \cap C)$, and $\#(T \cap C)$.

However, don't forget the portion that is in all three overlaps. When we added up $\#P$ + $\#T$ + $\#C$, this part was TRIPLE counted (there were TWO overlaps on it). When we subtracted $\#(P \cap T)$, this eliminated one overlap. When we subtracted $\#(P \cap C)$, this eliminated another overlap, and we should have stopped here, because we eliminated the two overlaps. However, we had to also subtract $\#(T \cap C)$ - this means we subtracted one too many times. Therefore, we had to add back this amount we subtracted, which is why we add $\#(P \cap T \cap C)$.

2. Let $A = \{E, H, J, T, S, G, Z, N\}$, $B = \{L, Y, U, I, O, Z, N, R, K\}$, $C = \{G, Q, N, F, R, K, Y, F, Z, S, H, J, T\}$ and $D = \{U, I, N, G, F, X, R, Z, K, V\}$:

   (a) Redraw the 4-set Venn Diagram shown to the right, and place all the elements in their proper section.

(b) How many of the sections in the diagram to the right represent the intersection of 2 sets? 3 sets? All 4 sets?
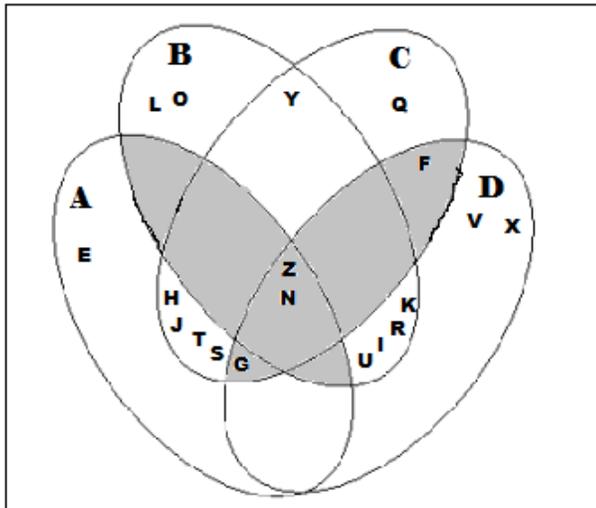
2 sets: 6 representations)
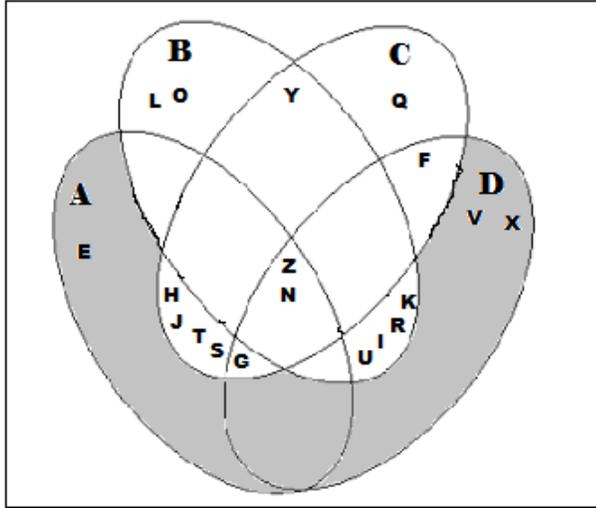
3 sets: 4 representations)

1 set: 1 representation

(c) What elements are in the set $(A \cap B) \cup (C \cap D)$? Shade this in on a Venn Diagram.

$(A \cap B) \cup (C \cap D) = \{F\}$

(d) What elements are in the set $(\overline{B} \cap \overline{C}) \cap (A \cup D)$? Shade this in on a Venn Diagram.

$(\overline{B} \cap \overline{C}) \cap (A \cup D) = \{F, Z, C, N\}$

3. *** Using the diagram in the previous question, write the Inclusion-Exclusion Principle for $|A \cup B \cup C \cup D|$.

$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| - |A \cap B \cap C \cap D|$