LINEAR RECURRENCE RELATIONS: HANDOUT 2

Exercise 1. For each recurrence relation, find the unique root, r, of the characteristic equation, and find a closed form for the sequence with $a_0 = 0$ and $a_1 = r$.

(1) $a_n = -2a_{n-1} - a_{n-2}$
(2) $a_n = 4a_{n-1} - 4a_{n-2}$

Exercise 2. Find a closed form for each of the following recurrence relations:

(1) $a_n = 6a_{n-1} - 9a_{n-2}; a_0 = 2, a_1 = 3$
(2) $a_n = 2a_{n-1} - a_{n-2}; a_0 = 5, a_1 = 3$
(3) $a_n = -4a_{n-1} - 4a_{n-2}; a_0 = 1, a_1 = -1$

Exercise 3. Have fun!

(1) Consider the recurrence relation $a_n = \frac{3}{2}a_{n-1} - \frac{1}{2}a_{n-2}$. Choose any starting values you like, a_0 and a_1. Before computing the sequence, compute $2a_1 - a_0$. Now find the first ten or so terms of the sequence. You may want to use a calculator. Explain what you notice.
(2) The number $3 + 2\sqrt{2}$ is approximately 5.82842712474619009. It is irrational, so it never repeats or terminates. Using your calculator, look at the decimal expansion of $(3 + 2\sqrt{2})^n$ for $n = 2, 3, 4, 5$, and maybe a few more if you like. You should notice that they get very close to taking on integer values. Try to explain why this happens. Here are a few hints:
    - Find a recurrence relation whose closed form is $a_n = (3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n$.
    - Look at the value of $(3 - 2\sqrt{2})^n$ for a few n. What do you notice?
(3) Recall the recurrence relation defined by $a_n = 2a_{n-1} + a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$. Compute the first few terms. Now compute the first few terms of the sequence $a_n = 6a_{n-1} - a_{n-2}$ with $a_0 = 1$ and $a_1 = 3$. Look closely at the two sequences and explain the connection. (Hint: Solve both recurrence relations, trying to find a connection between the roots.)
(4) One of the first examples we did was the recurrence relation $a_n = -a_{n-1} - a_{n-2}$. Pick any a_0 and a_1 you like, and compute the first few terms of the sequence. You should notice a pattern. Explain it! (Hint: Look at powers of the roots of the characteristic equation.)

(5) In the very first lecture, we saw that if f_n is the nth term of the Fibonacci sequence, that f_{n+1}/f_n gets very close to the value $\frac{1+\sqrt{5}}{2}$. By similar ideas from (2), explain why this happens.

(6) Compute the first few terms of $a_n = 3a_{n-1} - 2a_{n-1}$ with a_0 and a_1 chosen to your liking. Find a recurrence relation that is every third term of the one you just created. That is, the sequence generated is $a_0, a_3, a_6, a_9, \dots$.

(7) Consider the *third order* linear recurrence relation $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ with $a_0 = 3$, $a_1 = 2$, and $a_2 = 6$. Try to find a closed form for this sequence using the same type of technique we have already seen.

(8) In this exercise, you will find rational approximations to $\sqrt{3}$. That is, rational numbers that are very close to $\sqrt{3}$. To do this, find a recurrence relation with closed form $a_n = (1+\sqrt{3})^n + (1-\sqrt{3})^n$. Now figure out what happens to the $(1-\sqrt{3})^n$ part as n gets large. What do you expect to happen to a_n/a_{n-1} for large n? Now compute $(a_n/a_{n-1} - 1)^2$ for a few n and look at the decimal expansion.

(9) BONUS Pick any integer, k, that isn't a square. Find a recurrence relation and an integer c such that the sequence has the property that $a_n/a_{n-1} - c$ is a very good approximation of $\sqrt{k}$ for large n. (This is hard)