University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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Probability

1. In Major League Baseball, one of the most important probabilities measured is whether or not a player will make a hit. This probability is calculated using their batting average, which is collected over the course of each season. What type of probability is this (theoretical or experimental)?

   This is experimental probability, because we would have to record the number of hits a batter makes versus the number of times he is at bat over a period of time.

2. One die is rolled three times, with each roll being recorded.

   (a) How many possible outcomes are in this sample space?

   We calculate the total number of possibilities in the Sample Space by using counting techniques. For the first roll there are 6 outcomes, and for each of these outcomes there are 6 more for the second roll. For each of these 36 (6 * 6) two-roll combinations, there are 6 more possible outcomes for the third roll. Thus there are:

   $$6 * 6 * 6 = 216 \text{ total possible outcomes.}$$

(b) Let **D** be the event that the values of the three rolls are increasing by 1 from least to greatest ({1,2,3}, {2,3,4}, {3,4,5}, {4,5,6}). Let **E** be the event that the values of the three rolls are decreasing by 1 from greatest to least ({6,5,4}, {5,4,3}, {4,3,2}, {3,2,1}). Find $P(D \cup E)$ - the probability of event **D** or **E** occurring.

The occurrences for event **D** are:

$$D = \{ \{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{4,5,6\} \}$$

The occurrences for event **E** are:

$$E = \{ \{6,5,4\}, \{5,4,3\}, \{4,3,2\}, \{3,2,1\} \}$$

Noting that these two events are mutually exclusive (none of the occurrences of event **D** are also occurrences of event **E**), we can add the sum of the number of occurrences of these events to give us the number of occurrences of $P(E \cup D)$. This sum is 8, so

$$P(D \cup E) = \frac{8}{216} = \frac{1}{27} \approx 0.04 = 4\%$$

3. A coin is flipped 3 times. Below is an incomplete tree diagram that represents half of the corresponding Sample Space. Complete the tree diagram then find the total number of possibilities within the Sample Space.

Same as the tree diagram in the lesson. Since this is half of the sample space, we count the number of formations this tree gives us, then multiply it by 2. So there are 4 * 2 possible outcomes in our sample space.

4. Which of the following are not probabilities? Explain.

(a) 105%

(b) 1

(c) 0.65

(d) -0.45

(e) 0

The probability of any event **A** follows the inequality:

$$0 \leq P(A) \leq 1$$

$$(P(A) \text{ has to be in between } 0\% \text{ and } 100\%)$$

This means that a probability can equal zero, but it cannot be less than zero. It also means that a probability can equal 1 (or 100%), but it cannot be more than 1. For **a**, notice that if P(A) = 105% = 1.05, then the above inequality is not met, as $1.05 > 1$. Also for **d**, notice that if P(A) = -0.45, then the above inequality is not met, as $-0.45 < 0$.

5. A pair of dice is rolled and we record the two-dice combinations.

   (a) What is the probability that both dice will show a one?

   Let **A** be the event that the first die rolls a one. Let **B** be the event that the second die rolls a one.

   We are looking for $P(A \cap B)$. Since these two events are independent we get:

   $$P(A \cap B) = \text{P(A) * P(B)} = \frac{1}{6} * \frac{1}{6} = \frac{1}{36}$$

   (b) * Two dice are rolled again. One die shows a one, but the other die rolls under the table and is now out of sight. What is the probability that both dice will show a one, considering you have already gotten the first one?

   Since we already know the the first die successfully rolled a "1", we now only need to calculate the probability of a single die rolling a "1". Thus the probability is $\frac{1}{6}$.

6. You draw two cards from a standard deck of 52 cards.

   (a) What is the probability you will draw two hearts?

   Using counting, we know our Sample Space has 52 * 51 possible outcomes. For the event that we draw two hearts, there are 13 * 12 occurrences of that event. Thus, the probability is:

   $$P(\text{drawing two hearts}) = \frac{13 * 12}{52 * 51} = \frac{156}{2652} = \frac{1}{17} \approx 0.06 = 6\%$$

(b) * What is the probability you will draw two cards of the same suite?

In this case, since it does not matter what suite the first card is, we are only focused on the probability that the second card will be the same suite as the first. Since there would only be 12 cards left of the suite we drew for the first card, and 51 cards left in the deck, the probability is:

$$P(\text{drawing the same suite}) = \frac{12}{51} = \frac{4}{17} \approx 0.24 = 24\%$$

We also could have multiplied the probability in part **a** by 4, considering there would have been 4 suites to choose from:

$$P(\text{drawing the same suite}) =$$
$$\frac{4 * 13 * 12}{52 * 51} = \frac{52 * 12}{52 * 51} = \frac{52}{52} * \frac{12}{52} = 1 * \frac{12}{51} = \frac{4}{17} \approx 0.24 = 24\%$$

7. We roll a pair of dice. If **A** is the event such that the sum of the dice is even, and **B** is the event such that at the sum of the dice is 6, then find $P(A \cap B)$. (Hint: you only need to find the probability of one of these events occurring)

We first note that there are 6 * 6 = 36 total possible outcomes in this Sample Space.

Now, because any even number we roll must also be a 6, we only need to find the probability of rolling a 6! Thinking of all the different two-dice combinations that add to 6 when rolling dice, we get this set of occurrences:

{{1,5}, {2,4}, {3,3}, {4,2}, {5,1}}

Thus,

$$P(A \cap B) = P(B) = \frac{5}{36} \approx 14\%$$

*If you are not convinced by this solution, write out all the two-dice combinations for both of these events and find the occurrences that are in both set of occurrences.

8. There are 6 red balls, 8 blue balls, and 7 green balls in a box.

   (a) If one ball is randomly drawn from the box, what is the probability that the ball will not be red or blue.

   This is the same as finding the probability that the selected ball will be green. There are 7 green balls and 21 balls in total, therefore the probability is:

$$P(\text{not drawing red or blue}) = P(\text{drawing green}) = \frac{7}{21} = \frac{1}{3} = 33\%$$

   (b) * Now let's consider if the balls are numbered from 1 to 21 with the first 1 to 6 being red, 7 to 14 being blue, and 15 to 21 being green. If three balls are selected, what is the probability of event **X**, such that the values of the balls are increasing by double with each selection ({1,2,4}, {2,4,8}, {3,6,12}, ...). Once a ball is drawn, it cannot be drawn again.

   First let's find the total number of possible outcomes in the Sample Space. There are 21 balls in total and 3 balls are being drawn, and once a ball is drawn it cannot be drawn again. So we have:

$$21 * 20 * 19 = 7980 \text{ possible outcomes}$$

   To find the different occurrences for the event **X**, let's start with occurrence where the first ball drawn is 1, then find the occurrences where the first ball drawn is 2, and so on.

$$\{1,2,4\}$$

   This is the only occurrence we have where the first ball drawn is 1. It is similar for all other cases.

$$\{1,2,4\}$$
$$\{2,4,8\}$$
$$\{3,6,12\}$$
$$\{4,8,16\}$$
$$\{5,10,20\}$$

   If we were to keep going we would get: {6,12,24}, but there are only 21 balls in the box! So, we see that there are only 5 occurrences for this event, so the probability is:

$$P(X) = \frac{5}{7980} = \frac{1}{1596} \approx 0.00006 = 0.06\%$$

Conditional Probaility

9. If P(A) = 10%, P(B) = 45%, and $P(A \cap B) = 5\%$, find $P(A \mid B)$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.05}{0.45} \approx 11\%$$

10. Given that events **A** and **B** are mutually exclusive, without performing any calculations, find $P(A \mid B)$.

Since the two events are mutually exclusive, we know their intersection is equal to 0; therefore the answer to this question is also 0. We would get

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{P(B)} = 0$$

11. You are the teacher of a classroom and you are going over the results of two tests. You have found that 30% of your students passed both tests and 45% of them passed the first test. What percent of students that passed the first test also passed the second one.

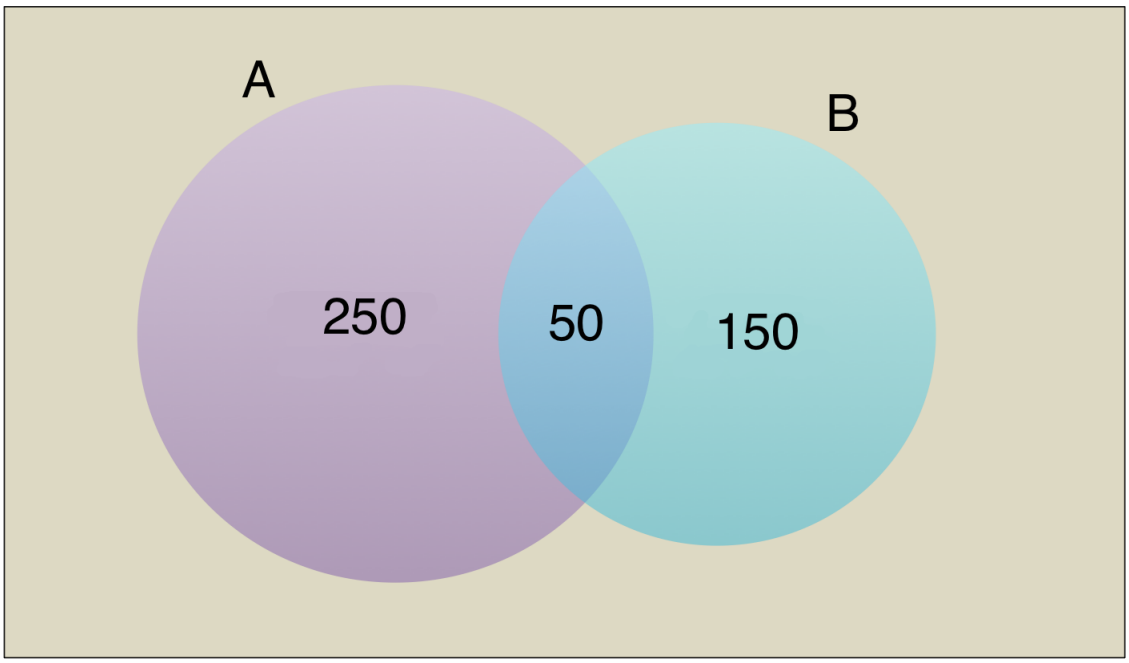
Let **A** be the event that a student passed the first test and **B** be the event that a student passed the second test. We are given that P(A) = 45% and $P(A \cap B) = 30\%$, and we need to find $P(B \mid A)$. It is done as follows:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{0.30}{0.45} = 0.66 = 66\%$$

12. * There are a total of 500 credit card owners. 300 of these owners are with Visa, 200 of these owners are with Mastercard, and 50 of them are with both.

(a) Introduce events **A** and **B**, then construct a Venn Diagram to represent this information. Be sure to include the number of owners in each respective circle.

Let **A** be the event that a card owner is with Mastercard. Let **B** be the event that that a card owner is with Visa. Then the Venn diagram looks like this:



(b) Given that a random card owner is with Mastercard, what is the probability they are also with Visa.

We are looking to find $P(A \mid B)$. We know that $P(B) = \frac{200}{500} = \frac{2}{5} = 0.4$ and $P(A \cap B) = \frac{50}{500} = \frac{1}{10} = 0.1$, therefore,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.1}{0.4} = 0.25 = 25\% = \frac{1}{4}$$

13. * You are playing a game of "Blackjack" against a dealer. If two, random cards each are dealt to you and the dealer in, alternating order (you are dealt the first card, they are dealt the second), what is the probability that the sum of your two cards is 21? You are given that:

- All face cards (King, Queen, Jack) have a value of 10
- All numbered cards have a value that is the same as their number (5 has a value of 5)
- The Ace has a value of 11.
- The dealer has neither a card with value 10 or 11 (no face cards or aces).

First let's find the number of total possibilities your two card hand can take. There are 52 possibilities for your first card, then 50 possibilities for the second card (The 50 comes from the alternating order of dealing). Thus there are:

52 * 50 = 2600 total possible 2 card hands

We now must calculate the amount of occurrences where the sum of your 2 card hands is 21. There are four cards with a value of 10 (King, Queen, Jack, 10) and these cards can appear in 4 different suites; therefore there are 16 possibilities for the first card to have a value of 10. As for the second card we require an Ace, which there are 4 of for each of the 16 cards with a value of 10. So we have:

16 * 4 = 64 different hands that add to 21.

However, we are still not done, as we have not considered if the first card we received was an Ace. We have looked at the following types of hands: {KA, QA, JA, TA} (not considering the suite). Thus if we swap the order for each Ace we get: {AK, AQ, AJ, AT}, it is then clear that there are double the amount of occurrences we initially had. So the probability that you will be dealt a hand that adds up to 21 is:

$$\frac{2 * (16 * 4)}{52 * 50} = \frac{128}{2600} = \frac{16}{325} \approx 5\%$$