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Grade 11/12 Math Circles

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Recurrences, Part 3

Definition of an L-system

An *L-system* or *Lindenmayer system* is a parallel rewriting system.

By parallel, we mean we each step involves replacing every possible occurrence at the same time.

We also need to specify the starting condition/symbol.

Rewriting Rules

We use the rule

$$X \to Y$$

to mean "replace every occurrence of X with Y."

For example, if we have the rule $X \to ABX$ and we have the current word:

$$ABXBAX$$

then the next word will be

$$ABABXBAABX$$

What does this sequence do?

1. $A \to AB$
2. $B \to \epsilon$

(Here ϵ means "this disappears.")

Start with $A \;\to\; AB \;\to\; AB \;\to\; AB$

Not much.

This just generates AB at the first step, which becomes AB after the second step, and so on.

So, this is just the word AB.

This indicates that AB is a *fixed point* of this recurrence.

Definition: A *fixed point* of a function f is a value t such that $f(t) = t$.

Notice: if $f(t) = t$, then $f(f(t)) = f(t) = t$, which means $f^n(t) = f(f(\cdots f(t))\cdots) = t$.

What does this sequence do?

1. $A \to AB$
2. $B \to B$

Start with $A \;\to\; AB \;\to\; ABB \;\to\; ABBB \;\to\; \ldots$

After n iterations of the rewriting rules, we have the word

$$AB^n = ABB\cdots B.$$

What does this sequence do?

1. $A \to B$

2. $B \to AB$

Start with $A \;\to\; B \;\to\; AB \;\to\; BAB \;\to\; ABBAB \;\to\; BABABBAB$

Notice what happens when we find the sum of the number of occurrences of A and B.

Sequence	**A**	**B**	**A + B**
A	1	0	1
B	0	1	1
AB	1	1	2
BAB	1	2	3
ABBAB	2	3	5
BABABBAB	3	5	8

This forms a Fibonacci Sequence. Call the sum of the number of occurrences of A and B f_n for each term of the sequence.

Also, notice the ratio

$$\frac{|A|}{|B|} = \frac{f_{n-1}}{f_{n-2}}$$

Where $|A|$ and $|B|$ are the number of occurrences of A and B, respectively, in each term of the sequence.

$$\lim_{n\to\infty} \frac{f_{n-1}}{f_{n-2}} = \varphi$$

What does this sequence do?

1. $A \to ABA$
2. $B \to BBB$

Start with $A \;\to\; ABA \;\to\; ABABBBABA \;\to\; ABABBBABABBBBBBBBBABABBBABA$

This sequence forms what's called Cantor Dust.

Cantor Dust

The line at the top of the figure represents the first term in the sequence, A. The line below represents the second term in the sequence, ABA, where B represents a break in the line above it. This then recurses.

How much of the line is removed?

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$$

But this is just a geometric series with the first term being $a = \frac{1}{3}$ and the common ratio being $r = \frac{2}{3}$. This series is equal to

$$\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1 = \frac{1}{3}\sum_{n=1}^{\infty}(\frac{2}{3})^{n-1}$$

Therefore, this whole line will be removed.

Will a length of $\frac{1}{4}$ ever be removed for instance?

It will never be removed since $\frac{2}{9} < \frac{1}{4} < \frac{1}{3}$

What does this sequence do?

1. $X \to X + YF$
2. $Y \to FX - Y$

Start with $FX \;\; \to \;\; FX + YF \;\; \to \;\; FX + YF + FX - YF$
$\to \;\; FX + YF + FX - YF + FX + YF - FX - YF.$

Then we take away the Xs and Ys to get

F + F + F - F + F + F - F - F

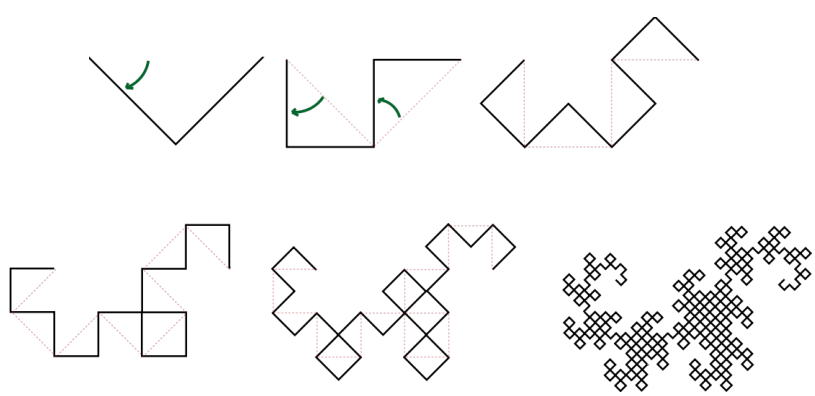
This sequence will form what's called a dragon curve.

Dragon Curve

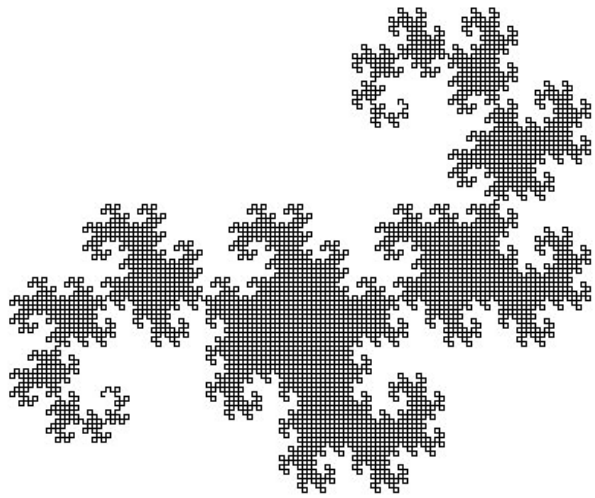
Do the following things:

- F means move forward
- $+$ means turn clockwise 90°
- $-$ means turn counterclockwise 90°

Here are some steps:



This is what the figure will look like after we perform many steps.



Another L-system

Start with:

$$F - -F - -F$$

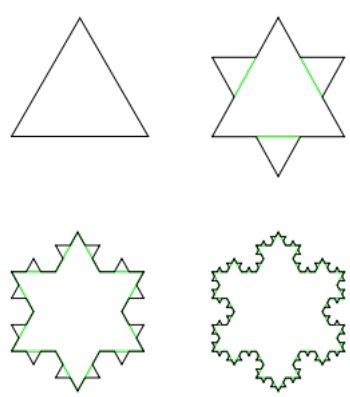
One rule:

$$F \to F + F - -F + F$$

Let:

- F: move forward
- $+$: move clockwise by $\frac{\pi}{3}$
- $-$: move counterclockwise by $\frac{\pi}{3}$

This forms what's called a **Koch Snowflake**:



What does this sequence do?

1. $0 \to 01$
2. $1 \to 10$

Start 0 $\to$ 01 $\to$ 0110 $\to$ <u>01101001</u> $\to$ <u>0110100</u>110010110.

This is called the Prouhet-Thue-Morse Sequence and it does lot of things.

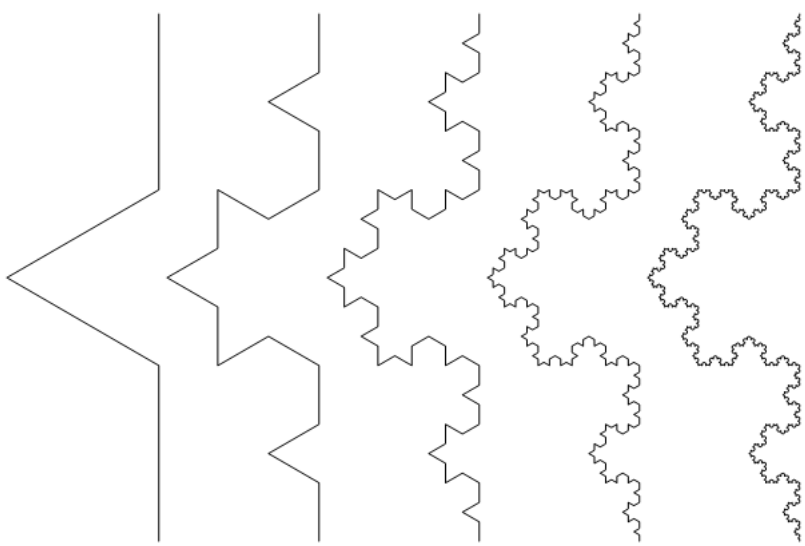
Prouhet-Thue-Morse Sequence

Call the sequence $\mathbf{t} = t_0 t_1 t_2 ... = 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ ...$

Draw a picture

- If $t_n = 0$, turn by π
- If $t_n = 1$, move ahead one unit and then rotate counterclockwise by an angle of $\frac{\pi}{3}$.

This forms what's called a **Koch Curve**



Another definition of the PTM Sequence

Write out all integers in base-2 (i.e., binary).

$$0,\ 1,\ 10,\ 11,\ 100,\ 101,\ 110,\ 111,\ 1000,\ 1001,...$$

Define $s_2(x)$ to the sum of the digits in the base-2 representation of x.

$$0\ 1\ 1\ 0\ 1\ 2\ 2\ 3\ 1\ 2...$$

If we did this for integers [0,16] keeping in mind that 2,4 = 0 and 3 = 1 in base-2 then we get.

$$0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0$$

Yet another definition of the PTM Sequence

Write the number, then flip the bits and rewrite it.

Define:

$$\begin{aligned} X_0 &= 0 \\ X_{n+1} &= X_n \bar{X}_n \end{aligned}$$

where $\bar{x}$ means change all 0's to 1's, and 1's to 0's.

$$0 \ \rightarrow\ 01 \ \rightarrow\ 0110 \ \rightarrow\ \underline{0110}\,\underline{1001} \ \rightarrow\ \underline{01101001}\ 10010110$$

Equivalent Definitions

Prove that the last two definitions are equivalent. In other words, prove:

$$t_n = s_2(n) \mod 2$$

where t_n is the nth term of X as defined on the previous slide.

Proof: Use strong mathematical induction.

Base cases:

If $n = 0$, $s_2(0) = 0 = t_0$.

If $n = 1$, $s_2(1) = 1 = t_1$.

Assume true for all $n < 2^k$, $k \geq 1$. That is $s_2(n) = t_n$ for all $n < 2^k$.

We will show the result holds for any n such that $2^k \leq n < 2^{k+1}$.

Now, t_n is the nth symbol of $X_{k+1} = X_k \bar{X}_k$.

Thus, t_n is the $(n - 2^k)$th symbol of $\bar{X}_k$, which means that $t_n = (t_{n-2^k} + 1) \bmod 2$.

By induction, we know that $t_{n-2^k} = s_2(n - 2^k) \bmod 2$.

Since $2^k \leq n < 2^{k+1}$, we have $s_2(n) = s_2(n - 2^k) + 1$, since the subtraction of 2_k will remove the leading 1 in the base-2 representation, but keep all other digits the same.

Thus,

$$\begin{aligned} t_n &= (t_{n-2^k} + 1) \bmod 2 \\ &= (s_2(n - 2^k) + 1) \bmod 2 \\ &= ((s_2(n) + 1) + 1) \bmod 2 \\ &= s_2(n) \bmod 2 \end{aligned}$$

which proves the result.

Morphism

A *morphism* is a map h on strings that satisfies the identity

$$h(xy) = h(x)h(y)$$

for all strings x, y.

PTM morphism

Define the PTM Morphism

$$\mu(0) = 01,$$

$$\mu(1) = 10.$$

Then

$$\begin{aligned} \mu(0) &= 01 \\ \mu^2(0) = \mu(\mu(0)) &= 0110 \\ \mu^3(0) &= 01101001 \\ \mu^4(0) &= 0110100110010110 \end{aligned}$$

Fixed Point of a Morphism

The *fixed point of a morphism h beginning with a* is:

$$h^\omega(a) = \lim_{m\to\infty} h^m(a).$$

Equivalent Definitions

Let's prove $\mu^n(0) = X_n$.

Use mathematical induction.

Base case: $n = 0$. $\mu^0(0) = 0 = X_0$.

Assume the result holds for $n = k$. That is, $\mu^k(0) = X_k$.

Prove true for $n = k + 1$.

$$\begin{aligned}
\mu^{k+1}(0) &= \mu^k(\mu(0)) \\
&= \mu^k(01) \\
&= \mu^k(0)\mu^k(1) \\
&= X_k\bar{X}_k \\
&= X_{k+1}
\end{aligned}$$

Overlap-Free

Let x be any word

A *square* is a word of the form xx.

For example:

murmur, mama, papa, dada, hotshots

A *cube* is a word of the form xxx.

For example:

lalala

An *overlap* is a word $axaxa$, with a single letter a and x being any word.

For example, if $x =$ ma and $a =$ a, then

"mamam" is an overlap

How does this relate to the Thue-Morse sequence?

The Thue-Morse sequence is overlap-free.

Last Problem

Consider the sequence:

$$\frac{1}{2}, \frac{1/2}{3/4}, \frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}}, \frac{\frac{\frac{1/2}{3/4}}{\frac{5/6}{7/8}}}{\frac{\frac{9/10}{11/12}}{\frac{13/14}{15/16}}}$$

What does this converge to?

First, observe the limit is:

$$\Pi_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$$

Let:

$$P = \Pi_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}}$$
$$Q = \Pi_{n\geq 0}\left(\frac{2n}{2n+1}\right)^{(-1)^{t_n}}$$

Notice that

$$PQ = \frac{1}{2}\Pi_{n\geq 1}\left(\frac{n}{n+1}\right)^{(-1)^{t_n}}$$

$$PQ = \frac{1}{2}\Pi_{n\geq 1}\left(\frac{n}{n+1}\right)^{(-1)^{t_n}}$$

Rewrite, breaking over odd and even indices:

$$\begin{aligned} PQ &= \frac{1}{2}\Pi_{n\geq 0}\left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_{2n+1}}} \Pi_{n\geq 1}\left(\frac{2n}{2n+1}\right)^{(-1)^{t_n}} \\ &= \frac{1}{2}P^{-1}Q. \end{aligned}$$

Thus, $P^2 = \frac{1}{2}$.