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Grade 6 Math Circles

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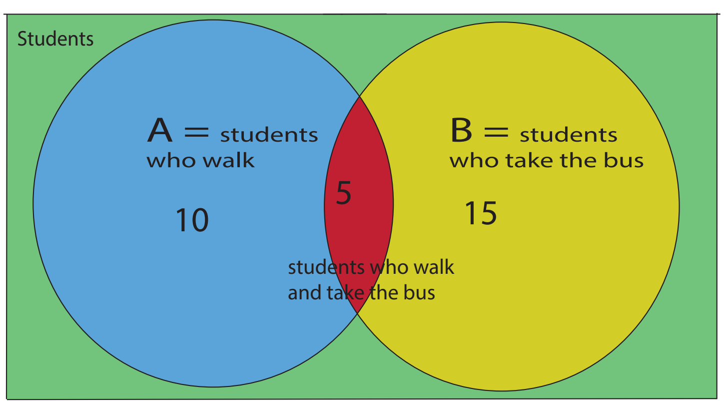
Counting

What is Counting?

When you think of the word counting, you probably think of counting numbers like you learned in kindergarten: 1,2,3,...
The counting that we will learn today is more complex, but not altogether much different. We will be counting the number of different ways a certain event can occur, while exploring some real-life counting problems. Finally, we will learn some new concepts that can be applied to more easily solve these problems.
First, let's do a quick review:

Venn Diagrams

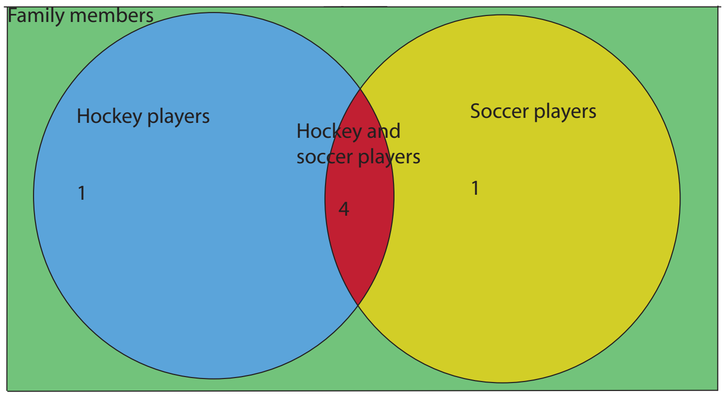
A and B both represent sets. The part where the circles overlap represents all of the elements that A and B have in common.



Example: Mr. Ratburn's class has 30 students. 10 students walk to school, 15 take the bus, and 5 walk and take the bus. Fill in the Venn diagram above. What should the sets A and B represent?

In a family of 6, everyone plays soccer or hockey. 4 members play both sports and 1 member plays only hockey. How many family members only play soccer? Use a Venn Diagram to show your answer.

6 - 4 - 1 = 1 family member who only plays soccer.



A First Counting Example: Baking a Cake

You are making a birthday cake and you have lots of choices:

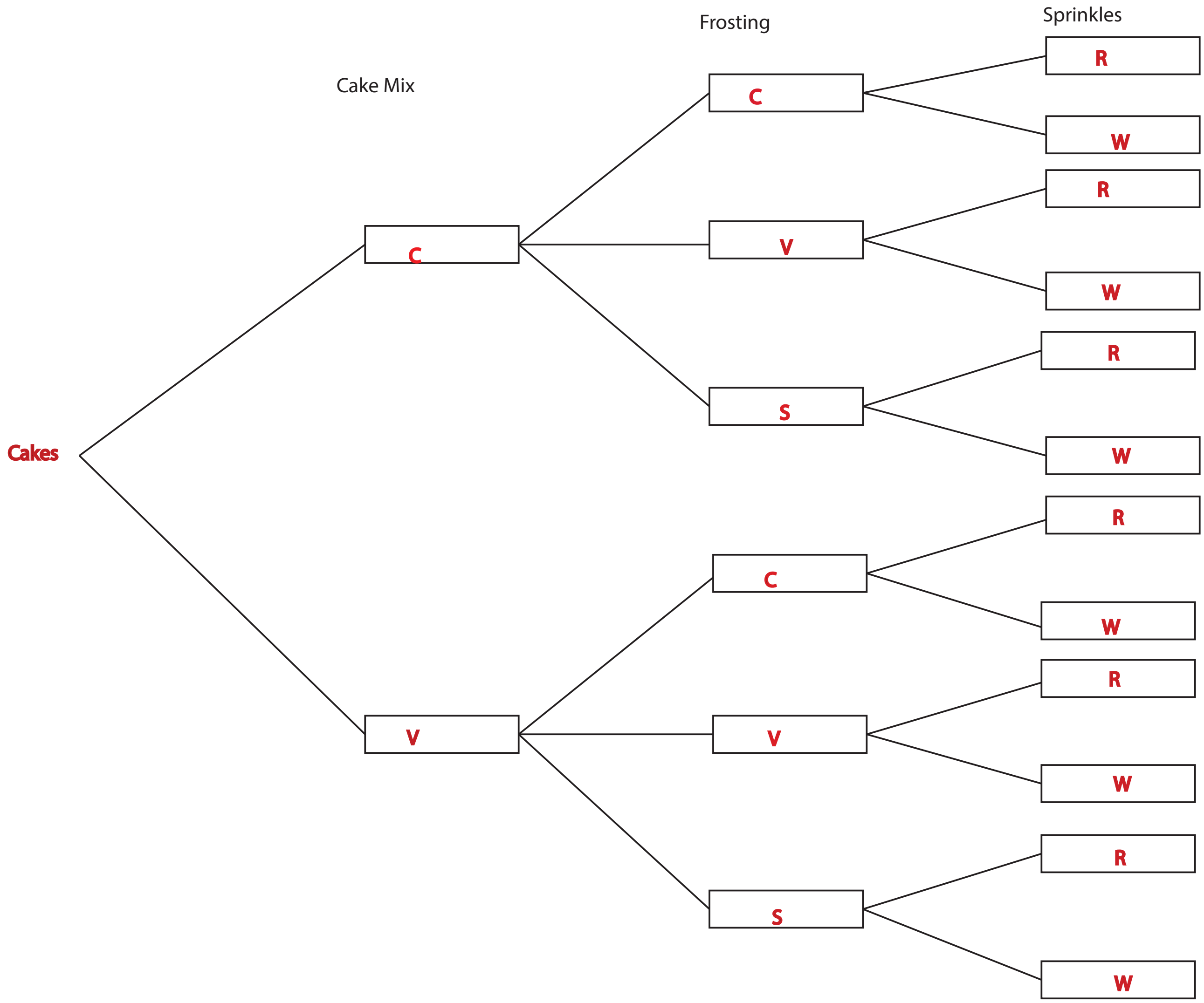
- 2 choices of cake mix: Chocolate or Vanilla
- 3 choices of frosting: Chocolate, Vanilla or Strawberry
- 2 choices of sprinkles: red or white

How many different ways can you make the cake?

Using the abbreviations C for chocolate, V for vanilla, S for strawberry, R for red and W for white, list all the possible combinations of cakes:

{C,C,R}, {C,C,W}, {C,V,R}, {C, V, W}, {C, S, R}, {C, S, W}, {V, C, R}, {V, C, W}, {V,V,R}, {V, V, W}, {V, S, R}, {V, S, W}

We can solve this problem more easily using a *tree diagram*. By following each path, we can list all the different possibilities of cakes that we can bake.



The first 2 boxes show that we have 2 possibilities of cake mix: chocolate or vanilla.
Following either path, we have 3 possibilities of frosting: chocolate, vanilla or strawberry.
Once we decide which frosting we want, we have 2 possibilities for sprinkles: red or white.
To figure out how many possibilities of cakes we can bake, we simply count the number of boxes (the sprinkles in this case).
So, there are 12 different cakes we can make.

The Fundamental Counting Principle

What if there was an easier way to solve the birthday cake problem than just writing out all the different combinations in a tree?

It turns out there is, and it is all based on the following:

The Fundamental Counting Principle says that when there are **m** ways of doing one thing and **n** ways of doing another thing, then there are **m x n** ways of doing both things.

What does this mean? Let's return to the birthday cake example:

We had 3 different "things" to look at: cake mix, frosting, and sprinkles. Now we need to relate these "things" to our counting principle so that we can find the number of ways we can do each "thing." There were 2 ways of choosing cake mix: chocolate or vanilla.

There were 3 ways of choosing frosting: chocolate, vanilla or strawberry.

There were 2 ways of choosing sprinkles: red or white.

Using the *Fundamental Counting Principle*, we can see that there are 2 x 3 x 2 = 12 ways to bake the cake.

Is this the same answer that we got from using the tree diagram?

Yes Would the answer change if we considered sprinkles first, then the cake mix and finally frosting? Why?

No, the answer would not change. The order does not matter since we are still choosing each item the same number of ways, and even if we choose what kind of sprinkles we want first, we still have the same number of options for frosting and cake mix.

Some More Examples

Now let's try a few more examples similar to the birthday cake problem:

1. Suzy has to choose an outfit for school tomorrow. She has 3 pairs of shoes to choose from, 4 shirts to choose from and 2 pairs of pants to choose from. How many different possible outfits can she wear?

3 shoes $\times$ 4 shirts $\times$ 2 pants = 24 outfits

2. You go to Build-a-Bear and can't decide which type of bear you want. There are 5 types of bears, 2 types of stuffing (lots or little), 2 different sounds you can put inside your bear and 5 different outfits. How many combinations of bears exist?

5 bears $\times$ 2 stuffings $\times$ 2 sounds $\times$ 5 outfits = 100 combinations

3. Arnold goes to the gym and keeps his clothes in a locker. The lock has a 4-digit passcode with the digits between 0 and 9. How many possible combinations are there?

$10 \times 10 \times 10 \times 10 = 10\,000$ combinations

Permutations: Order!

In the past few examples, the order with which we found the answer did not matter. For example, in the birthday cake problem, it didn't matter if we figured out that we have 2 possibilities of sprinkles before or after we figured out that we have 3 combinations of frosting to choose from.

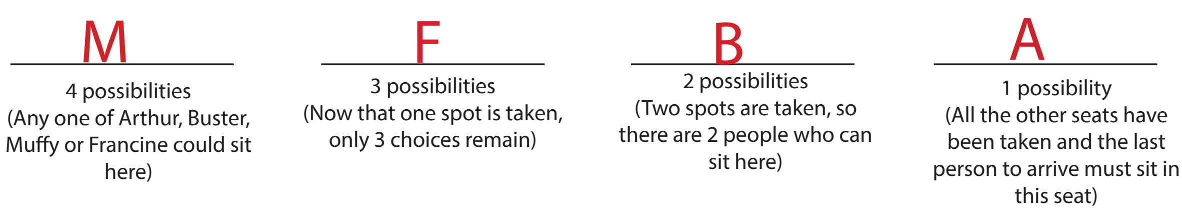
In the next set of examples, we will look at problems where the order with which we choose something matters.

Arthur, Buster, Francine and Muffy go to the movie theatre to see *The Lego Movie* and sit beside each other in a row. How many different ways can they be seated?

First, let's try to list all of the different ways the 4 friends can be seated:

{ABFM, ABMF, AFBM, AFMB, AMFB, AMBF, BAFM, BAMF, BMAF, BMFA, BFAM, BFMA, FABM, FAMB, FBMA, FBAM, FMAB, FMBA, MABF, MAFB, MFBA, MFAB, MBFA, MBAF}

Now for an easier way to solve the problem! Let's visualize the seats:



When the 4 friends arrive at the theatre, there are 4 seats available. So, we can put any one of them in the first seat. There are 4 possibilities for that first seat (Arthur, Buster, Muffy, Francine). Let's have Muffy take the first seat.

Then for the second seat, we have 3 possibilities, since Muffy is already seated. One of Arthur, Buster or Francine can sit in this seat. Let's have Francine take the second seat.

Now there's only Arthur and Buster left, and 2 seats left. So, there are 2 possibilities. Let's have Buster take this seat.

With only 1 seat left, Arthur must take this seat: there are no other possibilities, since everyone else is already seated.

$\underline{\quad 4 \quad}$ x $\underline{\quad 3 \quad}$ x $\underline{\quad 2 \quad}$ x $\underline{\quad 1 \quad}$

To count the number of possibilities, we will again use the *Fundamental Counting Principle*. We have 4 ways of choosing who sits in the first seat, followed by 3 ways for the second seat, 2 ways for the third seat and 1 way for the fourth seat.
So, we have 4 x 3 x 2 x 1 = 24 possible seating arrangements.

When we find possibilities that are arranged in order, like the example at the movies, we find a **permutation**.
When we find possibilities where order doesn't matter, like the birthday cake example, we find a
combination.

Some More Examples

Let's try a few more ordered (permutation) examples, similar to the movie problem:
1. A class of 10 students must do oral presentations, and the students must pick from a hat the order in which they will present. How many presentation schedules are possible?
$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3\,628\,800$
2. There are 8 speed skaters in the Olympic final representing the following countries: Canada (C), USA (U), Republic of Korea (K), Japan (J), Netherlands (N), Russia (R), China (P) and Italy (I). Assuming there are no ties, how many different ways can gold, silver and bronze be awarded?
$8 \times 7 \times 6 = 336$

Factorials

What if we had 10 friends going to the movies instead of just 4?

Based on the calculations we made from above, we would have

10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 3 628 800 possible seating arrangements for the 10 friends.

But mathematicians are lazy! So instead of writing out all of this multiplication, we use **factorial** notation.

The factorial of a number is the product of all the positive whole numbers less than or equal to that number. We show factorial with an exclamation mark, !

So the factorial of any number n is $n! = n \times (n\text{-}1) \times (n\text{-}2) \times ... \times 3 \times 2 \times 1$

So, 10! = 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 3 628 800.

From the example with the movies above, 4! = 4 x 3 x 2 x 1 = 24.

The only weird rule to remember is that 0! = 1.

Let's try a few examples of factorials!

a) $2! = 2 \times 1 = 2$

b) $3! = 3 \times 2 \times 1 = 6$

c) $1! = 1$

d) $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Now let's try to solve a permutation question using factorial notation.

You have 6 different cookies that you are about to eat. How many different ways can you order the way you will eat the cookies?

$6! = 720$

Grouping

Arthur, Buster, Francine and Muffy go see *The Lego Movie* again because everything about it is awesome. This time, Arthur and Buster want to make sure they can sit together. How many arrangements of the 4 friends exist where Arthur and Buster are sitting together?

Try to list all of the ways that Arthur and Buster can sit together:

{ABMF, BAMF, ABFM, BAFM, MABF, MBAF, FABM, FBAM, MFAB, MFBA, FMAB, FMBA}

To solve this problem, we have to **group** Arthur and Buster together. We count them as one item, since they will be sitting together.

Then, we only have 3 places to give out: Arthur/Buster, Francine and Muffy.

Instead of having 4! possibilities, we have 3! possibilities, since there are only 3 places to decide.

But we must also remember that Arthur and Buster can sit in the order Arthur Buster or Buster Arthur.
So there are 2 possibilities for the way that Arthur and Buster sit within their "group."
Altogether, we have 3! x 2 = 12 ways to arrange the 4 friends so that Arthur and Buster can sit together.

Problems

1. You stop for dinner at a fast food restaurant on your way to Math Circles. Here are your burger options:

- white, whole wheat or cheese-flavoured bun
- chicken, beef or veggie burger
- Tomatoes, pickles, onions and lettuce as toppings

You are really in a rush and decide to only get one topping (you are also only getting one burger and one bun). How many possible burgers can be chosen?

2, There are 30 students in Mr. Johnson's class. If 16 only like Math, 3 like Math and English and 6 don't like Math or English, how many students only like English? (Use a Venn Diagram)

3. Should a permutation or combination be used in the following scenarios:
a) Selecting 20 students to go on a field trip
b) Assigning students their seat on the first day of school
c) Selecting what size of popcorn you want, whether or not you want butter on your popcorn, and which movie you want to see

4. Harry, Cedric, Fleur and Viktor have to face a dragon for the Triwizard Tournament.

They will each draw a number between 1 and 4 to determine which dragon they will face. How many different scenarios are there?

5. You have to pick a debating team with one boy from {Alain, Liam, Patrick} and one girl from {Michelle, Nicole, Karen, Lisa}. How many different teams can be formed?

6. A bag contains 5 balls: one blue, one yellow, one green, one red and one orange. If you draw 5 balls, how many possible arrangements exist if:
a) You keep the ball out of the bag after it is selected?(We call this "without replacement")
b) You put the ball back in the bag after each selection?(We call this "with replacement")

7. You roll a die 3 times and write down the 3 numbers in the order they appear. How many possible results are there?

8. It's time for qualification for the summer Olympics, and only 4 countries out of Canada (C), USA (U), Republic of Korea (K), Japan (J), Netherlands (N), Russia (R), China (P) and Italy (I) will qualify for the marathon. We need to determine who comes 1st, 2nd, 3rd and 4th (order matters). Assuming there are no ties, how many possible arrangements of

the top 4 exist?

9. A car's license plate consists of 4 letters followed by 3 numbers. Knowing that there are 26 letters to choose from and 10 numbers to choose from (0 to 9), how many possible license plates can be issued? (You do not need to find the number- a simplified answer is good enough)

10. Amy goes to the ice cream parlour where there are 20 different flavours. If she wants 2 scoops of different flavours, how many different ways can Amy order an ice cream cone?

11. 5 people are in a room for a meeting. When the meeting ends, each person shakes hands with each of the other people in the room exactly once. What is the total number of handshakes?

12. Arthur, Buster, Muffy and Francine return to see *The Lego Movie* a third time. This time, Arthur and Buster can't sit together, because they sing "Everything is Awesome" too loudly when they sit together. How many different ways can the 4 friends be seated if Arthur and Buster are not together? (Hint: Use the numbers calculated in the examples)

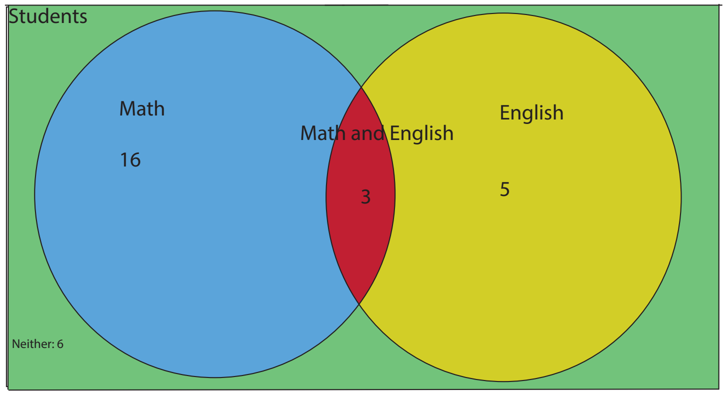
13. A hardware store sells single digits to be used for house numbers. There are five 5s, four 4s, three 3s and two 2s available. From this selection of digits, a customer is able to purchase their three-digit house number. Determine the number of possible house numbers this customer could form.

14. Permutations are formed using all of the digits 1,2,3,...,9 without repeating any numbers. Determine the number of possible permutations in each of the following cases (Answers may be left in factorial form):
a) even and odd digits alternate
b) the digits 1,2,3 are together but not necessarily in their natural order
c) 1 is before 9 but not necessarily right beside it

15. How many different ways can you order the letters of the word MATHEMATICS?

Solutions to Problems

1. $3 \times 3 \times 4 = 36$ burgers
2. 30 - 6 - 16 - 3 = 5 students like only English:



3. a) Combination
b) Permutation
c) Combination
4. $4! = 24$ scenarios
5. 12 teams
6. a) $5! = 120$ arrangements
b) $5 \times 5 \times 5 \times 5 \times 5 = 3125$ arrangements
7. $6 \times 6 \times 6 = 216$
8. $8 \times 7 \times 6 \times 5 = 1680$ arrangements
9. $26 \times 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 456\,976\,000$ license plates
10. $20 \times 19 = 380$ different ice cream cones
11. Each of the 5 people shake 4 hands. This gives 20 hands, but that counts every handshake twice (Person X shakes Person Y's hand AND Person Y shakes Person X's hand). So, the total number of handshakes is $20 \div 2 = 10$.
12. Total seating arrangements $= 4! = 24$
Total ways Arthur and Buster can sit together $= 3! \times 2! = 12$
So the total ways Arthur and Buster can not sit together = 24 - 12 = 12
13. There are 4 choices for a digit: 2, 3, 4, 5. Then we have $4 \times 4 \times 4 = 64$ possible house numbers. But if we have 2 and 2 for the first two digits, we will only have 3 choices for the last digit since there are only two 2s available. So, we have to take away one possibility, and there are 64 - 1 = 63 possible house numbers.
14. a) There are 5 odd numbers and 4 even numbers. So, we have $5!$ arrangements of the odd numbers and $4!$ arrangements of the even numbers. In total, there are $5! \times 4.$ arrangements.
b)(1,2,3) is a grouping, so we remove the numbers 1, 2, and 3 from the 9 original digits, and

add the grouping. We then have 9-6+1 = 7 "'numbers" to be arranged. But within (1,2,3), we can arrange these numbers 3! ways. So in total the number of arrangements is $7! \times 3!$.
c) There are 9! arrangements, but there are an equal amount of ways that 1 comes before 9 as the amount of ways that 1 comes after 9. So, there are $9! \div 2$ arrangements.
15. MATHEMATICS has 11 letters. So there should b 11! permutations of the word. However, we notice that A, T, and M are repeated. This means that we would obtain the same permutations twice using these letters. So, we have to divide by the permutations of repeated letters, to "cancel out" the repeated letters. We get $11! / (2! \times 2! \times 2!) = 4\,989\,600$