



Grade 6 Math Circles
March 10/11, 2015
Prime Time Solutions

Lights, Camera, Primes!

Today, we're going to talk about a special type of number. Whether this is your first or tenth time hearing about this type of number, there's always something new to learn. Welcome to Prime [Number] Time!

Warm-Up

Try to answer the following 7 questions in 3 minutes.

1. Identify the following numbers as either composite, prime, or neither:

- (a) 1 **neither**
- (b) 2 **prime**
- (c) 82 **composite**
- (d) 47 **prime**
- (e) 73 **prime**
- (f) 37 **prime**
- (g) 77 **composite**

2. Find the prime factorization of the following numbers:

- (a) 100 **$2 \times 2 \times 5 \times 5$**
- (b) 18 **$2 \times 3 \times 3$**
- (c) 73 **73**
- (d) 66 **$2 \times 3 \times 11$**

3. Find the GCD of the following pairs of numbers:

(a) 75 and 60 **15**

(b) 75 and 66 **3**

(c) 45 and 73 **1**

4. Find the LCM of the following pairs of numbers:

(a) 16 and 12 **48**

(b) 7 and 13 **91**

(c) 7 and 21 **21**

(d) 24 and 42 **168**

5. Is 45 319 divisible by 3? **No, since the sum of the digits is not divisible by 3.**

6. Is 725 128 divisible by 4? **Yes, since the last two digits form a number divisible by 4.**

7. Is 34 272 divisible 6? **Yes, since it is divisible by 2 and 3.**

Fundamental Knowledge

A *factor* of a number is a whole number that divides evenly into the other number. So, when you divide by a factor the remainder is zero. Factors come in pairs. When you divide a number by one of its factors, the answer or quotient is the paired factor.

A *prime number* is a number whose only factors are 1 and itself.

A *composite number* is a number with factors other than 1 and itself.

All numbers are either prime or composite, **except** for the number 1, which is neither.

Try it out:

1. What are all the factors of 24? **1, 2, 3, 4, 6, 8, 12, 24**

2. What are all the factors of 150? **1, 2, 3, 5, 6, 10, 15, 25, 30, 50, 75, 150**

3. Classify the following as prime or composite.

(a) 37 **prime**

(b) 52 **composite**

(c) 63 **composite**

Finding Primes

One (tedious) method of finding primes is called the **Sieve of Eratosthenes**:

1. Cross off 1 since it is not prime.
2. Find the first prime number (2) and circle it.
3. Go through the rest of the chart and cross off all of the multiples of two.
4. Go back to the beginning of the chart and find the first number that is not crossed off. Circle it, it is prime. Then go through the rest of the chart, crossing off the multiples of this number.
5. Repeat step 4, until every number in the chart is either circled or crossed off.
6. The numbers that are circled are the prime numbers.

Try it out:

Use this method to find all the prime numbers from 1 to 100. There should be 25 of them.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Prime Factorization

One way of describing numbers is by breaking them down into a product of their prime factors. This is called prime factorization. Every positive number can be prime factored. By definition the prime factorization of a prime number is the number itself, and the prime factorization of 1 is 1.

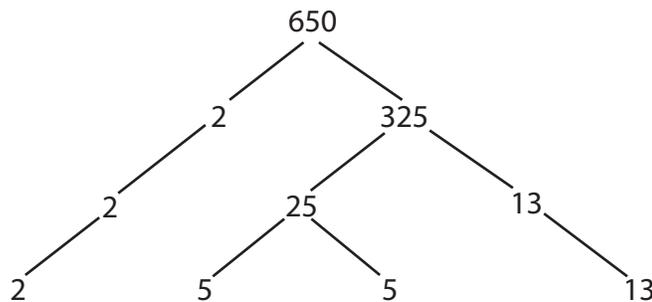
Prime factorizations are unique. Every number has only one prime factorization, and no two numbers share the same prime factorization.

Example:

Find the prime factorization of 650.

Solution:

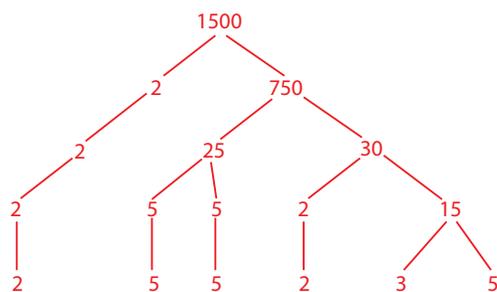
We create a factor tree.



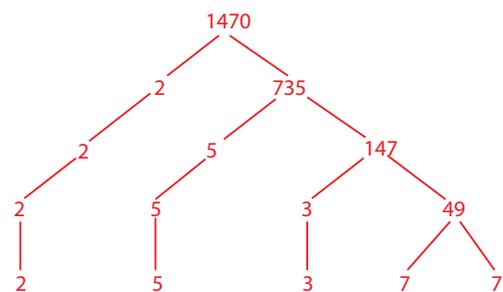
Thus we find that the prime factorization of 650 is $2 \times 5^2 \times 13$.

Try it out:

Find the prime factorization of 1500 and 1470.



$$1500 = 2 \times 2 \times 3 \times 5 \times 5 \times 5$$



$$1470 = 2 \times 3 \times 5 \times 7 \times 7$$

Greatest Common Divisor

The greatest common divisor or GCD of two or more numbers is the largest number that is a factor of each of the numbers. This can also be called the greatest common factor.

We're going to examine two different ways of finding the GCD:

Method 1

Find all the factors of each of the numbers and then identify the largest one that is a common factor of all the numbers. This is the GCD.

Method 2

Begin by finding the prime factorization of each of the numbers. Look at the prime factors of all the numbers and multiply the ones they have in common. This is the GCD.

Example: Find the GCD of 27 and 18.

Solution:

Using Method 1:

18: 1, 2, 3, 6, 9, 18

27: 1, 3, 9, 27

∴ the GCD is 9.

OR

Using Method 2:

$$18 = 2 \times 3 \times 3$$

$$27 = 3 \times 3 \times 3$$

∴ the GCD is $3 \times 3 = 9$.

Try it out:

Find the GCD of the following groups of numbers.

1. 36 and 64 4
2. 156 and 360 12
3. 328 and 208 and 400 8

Lowest Common Multiple

A *multiple* of a number is the product of the number and another whole number.

Factors and multiples are closely related. For example, 9 is a factor of 90, so 90 is a multiple of 9.

The least common multiple or LCM of two or more numbers is the smallest number that is a multiple of each of the numbers.

We're going to examine two different ways of finding the LCM:

Method 1

List the multiples of each number until you find the lowest one that is a common multiple of all the numbers. This is the LCM.

Method 2

Begin by finding the prime factorization of each of the numbers. Then multiply each factor the greatest number of times it occurs in any of the numbers. If the same factor occurs more than once in all the numbers, you multiply the factor the greatest number of times it occurs.

Example: Find the LCM of 27 and 18.

Solution:

Using Method 1:

18: 18, 36, 54

27: 27, 54

∴ the LCM is 54.

OR

Using Method 2:

$$18 = 2 \times 3 \times 3$$

$$27 = 3 \times 3 \times 3$$

∴ the LCM is $2 \times 3 \times 3 \times 3 = 54$.

Try it out:

Find the LCM of the following groups of numbers.

1. 12 and 18 **36**
2. 36 and 64 **576**
3. 50 and 75 and 9 **450**

Divisibility

When you're checking if a number is prime, you normally check to see if it's divisible by smaller numbers. You could answer questions about divisibility using long division, but mathematicians are lazy and long division would take too much time. Instead, we have discovered some neat tricks to speed up the process. Some of the simplest and most common can be found in the table below.

Divisible by:	Test:
2	The last digit in the number is even
3	The sum of all the digits is divisible by 3
4	The last two digits are divisible by 4
5	The number ends in 0 or 5
6	The number is divisible by 2 <i>and</i> 3
8	The last three digits are divisible by 8 Note: Can't use this rule for numbers less than 1000
9	The sum of the digits is divisible by 9
10	The number ends in 0
11	The <i>alternating</i> sum of the digits is divisible by 11 Notes: Start with subtraction. 0 is divisible by 11
12	The number is divisible by 3 <i>and</i> 4

Try it out:

1. Is 34 627 888 divisible by 8? **Yes, since the last three digits (888) are divisible by 8.**
2. Is 457 925 divisible by 9? **No, since the sum of the digits (32) is not divisible by 9.**
3. Is 6756 divisible by 12? **Yes, since it is divisible by 3 and 4.**

Theorems

Unique Factorization Theorem/Fundamental Theorem of Arithmetic

Any integer greater than 1 is either a prime number or can be written as a unique product of prime numbers (ignoring the order).

Infinite Primes Theorem

There are infinitely many primes.

Proof:

Suppose that $p_1 = 2 < p_2 = 3 < \dots < p_n$ are all of the primes. Let $P = (p_1 \times p_2 \times \dots \times p_n) + 1$. Let p be a prime dividing P . Then p can not be any of p_1, p_2, \dots, p_n , because if were, the remainder would be 1. So this prime p is still another prime, and p_1, p_2, \dots, p_n would not be all of the primes.

Wrap-Up

Today, you looked at prime numbers and their properties and uses. There are many more theorems based on the prime numbers. They are a fascinating part of mathematics, and a lot of research is still being done on this set of numbers.

Problem Set

Complete the following 15 problems. You may use a calculator.

1. Find the factors of the following numbers (use the divisibility tests when applicable)

(a) 96 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96

(b) 4276 1, 2, 4, 1069, 2138, 4276

(c) 89 1, 89

(d) 121 1, 11, 121

2. Find the prime factorization of the following numbers:

(a) 2459 2459

(b) 456 $2 \times 2 \times 2 \times 3 \times 19$

(c) 123 3×41

(d) 54320 $2 \times 2 \times 2 \times 2 \times 5 \times 7 \times 97$

3. Mr. Batman wants to split his class of 24 students into equal groups for a group project. List all of the possibilities of groups that Mr. Batman can divide his class into so that no students are left without a group.

The possibilities are: 1 group of 24 students, 2 groups of 12 students, 3 groups of 8 students, 4 groups of 6 students, 6 groups of 4 students, 8 groups of 3 students, 12 groups of 2 students, and 24 groups of 1 student.

4. Find the GCD of the following numbers:

- (a) 231, 646 **1**
- (b) 752, 48 **16**
- (c) 1024, 896 **128**
- (d) 378, 294 **42**
- (e) 375, 855 **15**
- (f) 54320, 5240, 2080 **40**

5. Find the LCM of the following numbers:

- (a) 21, 22 **462**
- (b) 92, 48 **1104**
- (c) 96, 56 **672**
- (d) 126, 294 **882**
- (e) 560, 40, 160 **1120**

6. The volume of a box is 1925cm^3 . What are the different possible dimensions of the box? (Note: the volume of a rectangular box is $length \times width \times height$.)

The different possible dimensions are:

$1\text{cm} \times 1\text{cm} \times 1925\text{cm}$

$1\text{cm} \times 35\text{cm} \times 55\text{cm}$

$1\text{cm} \times 5\text{cm} \times 385\text{cm}$

$5\text{cm} \times 5\text{cm} \times 55\text{cm}$

$1\text{cm} \times 7\text{cm} \times 275\text{cm}$

$5\text{cm} \times 7\text{cm} \times 55\text{cm}$

$1\text{cm} \times 11\text{cm} \times 175\text{cm}$

$5\text{cm} \times 11\text{cm} \times 35\text{cm}$

$1\text{cm} \times 25\text{cm} \times 77\text{cm}$

$7\text{cm} \times 11\text{cm} \times 25\text{cm}$

7. Using the divisibility tests, determine which numbers (2 to 11) are factors of the following:
- (a) 926 **2**
 - (b) 1420 **2, 4, 5, 10**
 - (c) 7848 **2, 3, 4, 6, 8, 9**
 - (d) 39208 **2, 4, 8**
8. What is the smallest number that you must multiply 48 by so that the product is divisible by 45? **15**
9. The eight digit number $1234\square678$ is divisible by 11. What is the digit \square ? **9**
10. What is the smallest prime number greater than 200? **211**
11. The four digit number $43\square\square$ is divisible by 3, 4 and 5. What are the last two digits? **20 or 80**
12. The product of three different positive integers is 144. What is the maximum possible sum of these three integers? **75 (72, 2, 1)**
13. Two campers have flashlights that blink every few seconds. The first camper's flashlight blinks every 4 seconds and the second's blinks every 6 seconds. If they turn on their flashlights at the same time, how many times will they flash simultaneously in one minute? **5 times**
14. James is buying burgers and buns. Burgers come in packs of 24, but buns come in packs of 16. What is the least number of packs of each that James will have to buy in order to have no burgers or buns left over? **2 burgers and 3 buns**
15. Two numbers are *co-prime* if they do not have any prime factors in common. List all possible numbers that are co-primes with the following numbers that are less than 50:
- (a) 390 **7, 11, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49**
 - (b) 210 **11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47**
 - (c) 49335 **2, 4, 7, 8, 14, 16, 17, 19, 23, 29, 31, 32, 34, 37, 38, 41, 43, 47, 49**