Math Circles
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Infinite Series
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NOTES

1 A Card Trick

Consider a standard deck of 52 cards stacked at the edge of a table. If you push the first one off the stack, how far will it go before it tips and falls on the floor? Half way, of course. Now lets start sliding the second one (and the first one). How far will those two go before they both fall? One-quarter card length. This gives a total overhang length of $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$. Push the third card as far as it goes (one-sixth of a card length), and now the total overhang length will be $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12}$. Can we ever push enough cards to make the total overhang length greater than one? At first glance it looks like 'no', but in fact the answer is **yes**, just push the fourth card!

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24}!$$

How far will they go? What if we have more cards? All these answers lie in the following table:

# Cards Pushed	Overhang Length
1	$\frac{1}{2}$
2	$\frac{3}{4}$
3	$\frac{11}{12}$
4	$\frac{25}{24}$
$\vdots$	$\vdots$
51	$\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{102} \approx 2.25$
$\vdots$	$\vdots$
100	≈ 2.5
$\vdots$	$\vdots$
1,000,000,000,000	≈ 14.1

Whoa hang on a second! We can push enough cards so that 14+ are hanging over? That's interesting. But whats more interesting is that it's not higher, after all we pushed *one-trillion* cards! That stack would reach the moon! Does this process ever end? Is there a limit to the overhang length? The answer is actually no! We can push enough cards to get

any overhang length! Let's go backwards: how many cards would we need to push to get an overhang length of 100?

$$\text{Answer: } \approx 10^{80}!$$

Thats roughly how many atoms are in the universe (so there are probably not enough cards). Let's see one more: an overhang length of *one-million*?

$$\text{Answer: } \approx 10^{567000} \ldots$$

so the number of cards we would need to push has over a five-hundred-thousand DIGITS! The point is that no matter how big an overhang length we want we can get it by pushing enough cards (we just might need to push a whole lot of them). The reason for this is that the numbers in the right column correspond to *half of the harmonic series*, we'll see what this is soon.

2 Introduction to Infinite Series

Definition 1. Let $a_0 + a_1, a_2, a_3, \ldots$ be an infinite collection of real numbers (i.e. a sequence of real numbers). An **infinite series** is an expression of the form

$$a_0 + a_1 + a_2 + \cdots = \sum_{n=0}^{\infty} a_n.$$

Examples:

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$
- $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$
- $\displaystyle\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots$

Our overall goal will be to learn what the 'sum' of an infinite series means and how to find it in some special cases (or how to show the sum is infinite). The first question we have to answer is: how do we sum infinitely many numbers? Let's first look at an example to build our intuition, and then the formal definition.

2.1 The Harmonic Series

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is called the **Harmonic Series**. This series *diverges*, that is, does not have a finite sum. Let's see a proof of that. There are many proofs of this fact but this one is my favorite.

Proposition 1. *The Harmonic Series diverges.*

Proof. Suppose for a contradiction that the harmonic series has a finite sum, say S. That is

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

But then we get

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \\ &= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \cdots \\ &> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \cdots \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots \\ &= S \end{aligned}$$

and so $S > S$ which is impossible for a finite real number. So S does not exist. □

Let's finally nail down a formal definition of what the "sum" of an infinite series is.

2.2 The Definition of "Sum"

Before we can talk about the sum of an infinite series we will need to understand *partial sums*:

Definition 2. If $\sum_{n=0}^{\infty} a_n$ is an infinite series, define the **sequence of partial sums**, $\{S_n\}$ to be $S_n =$ the sum of the terms up to n, so

$$S_n = a_0 + a_1 + \cdots + a_n.$$

Example: For $\sum_{n=1}^{\infty} \frac{1}{n}$,

$$S_1 = 1,\ S_2 = 1 + \frac{1}{2} = \frac{3}{2},\ S_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6},\ \text{etc.}$$

Now we can define the sum:

Definition 3. A series $\sum_{n=0}^{\infty} a_n$ **converges to S** or **has sum S** if $S_n \to S$ as $n \to \infty$. In other words, the S_n's get closer and closer to S as n gets larger and larger. If the S_n's get infinitely large or they don't get close to any finite real number then we say the series **diverges**.

Example: Consider $\sum_{n=0}^{\infty}(-1)^n$. This series has partial sums

$$S_1 = 1,\ S_2 = 0,\ S_3 = 1,\ S_4 = 0,\ \ldots$$

which clearly doesn't get close to one single number (the partial sums keep bouncing between 1 and 0). So $\sum_{n=0}^{\infty}(-1)^n$ *diverges.*

Note that it is not always easy to get a closed expression for S_n, and in some cases it's impossible! However, in the case of geometric series we always can.

3 Geometric Series

First let's try to guess the sum of the following series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

It looks like this series is adding up to 2! Each term adds half of the remaining distance from 0 to 2, so the end result should be 2. This is one example of a *geometric series*:

Definition 4. A **geometric series** is a series of the form

$$\sum_{n=0}^{\infty} ar^n$$

where $a, r \in \mathbb{R}$ and $a \neq 0$.

In this discussion we'll assume $r \geq 0$ but the same analysis works for negative r. The next question is: for which $r \geq 0$ does this series converge? Let's get an expression for the sequence of partial sums! We need two cases:

Case 1: $r = 1$. In this case our series gets pretty simple:

$$\sum_{n=0}^{\infty} a = a + a + a + a + \cdots .$$

So our n^{th} partial sum is

$$S_n = a + a + a + \cdots + a = na.$$

So as $n \to \infty$, $S_n \to \pm\infty$ since $a \neq 0$. Therefore, the series *diverges* if $r = 1$.

Case 2: $r \neq 1$. Then

$$S_n = a + ar + ar^2 + \cdots + ar^n$$

and

$$rS_n = ar + ar^2 + \cdots + ar^{n+1}.$$

Subtracting these two expressions gives us

$$S_n - rS_n = a - ar^{n+1}$$

which means

$$S_n = \frac{a - ar^{n+1}}{1-r}.$$

So what happens as $n \to \infty$? If $0 \leq r < 1$ then $r^{n+1} \to 0$, so

$$S_n \to \frac{a}{1-r}$$

which means the series *converges* to $\frac{a}{1-r}$ in this case.

On the other hand, if $r > 1$ then $r^{n+1} \to \infty$ as $n \to \infty$. This means that as $n \to \infty$ the partial sums S_n get infinitely large. So the series *diverges* in this case. All together we have shown

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} \text{ if } 0 \leq r < 1, \\ \text{diverges if } r \geq 1 \end{cases}.$$

Examples:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$, which we guessed from the start of this section. Here $a = 1$ and $r = \frac{1}{2} < 1$.

2. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1-\frac{2}{3}} = 3$. In this case $a = 1$ and $r = \frac{2}{3} < 1$.

3. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{5}{6} \cdot \left(\frac{5}{6}\right)^n = \frac{5}{6} \cdot \frac{1}{1-\frac{5}{6}} = 5$. In this case $a = \frac{5}{6}$ and $r = \frac{5}{6} < 1$.

4. $\displaystyle\sum_{n=0}^{\infty} 3\left(\frac{4}{3}\right)^n$ diverges since $r = \frac{4}{3} \geq 1$.

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}} = \sum_{n=0}^{\infty} \frac{8^n}{9^n} = \sum_{n=0}^{\infty} \left(\frac{8}{9}\right)^n = \frac{1}{1-\frac{8}{9}} = 9$. Here, $a = 1$, $r = \frac{8}{9}$.

4 Applications of Geometric Series

4.1 Expressing Repeating Decimals as Fractions

We can use the geometric series formula to express any repeating decimal as a single fraction! Let's see an example:

Example: Write $3.2131313\ldots = 3.2\overline{13}$ as a single fraction.

$$\begin{aligned}
3.2131313\ldots &= 3.2 + \frac{13}{10^3} + \frac{13}{10^5} + \frac{13}{10^7} + \cdots \\
&= 3.2 + \frac{13}{10^3}\left(1 + \frac{1}{100} + \frac{1}{100^2} + \frac{1}{100^3} + \cdots\right) \\
&= 3.2 + \frac{13}{10^3}\sum_{n=0}^{\infty}\left(\frac{1}{100}\right)^n \\
&= \frac{32}{10} + \frac{13}{1000}\left(\frac{1}{1-\frac{1}{100}}\right) \\
&= \frac{3181}{990}.
\end{aligned}$$

4.2 A Real-World Application

Suppose you are working for an interplanetary defense contractor and you set up a double-layered forcefield on a planet, to protect against laser fire. Each layer of the forcefield reflects $\frac{1}{2}$ of the incoming laser beam, absorbs $\frac{1}{4}$ of the beam, and transmits $\frac{1}{4}$ of the beam. If later on a ship fires a laser with intensity I at the planet, what fraction of I makes it through both shields?

Solution: We have to keep in mind that the initial one-fourth of the beam that gets through will start bouncing between the layers, so more than just $\frac{I}{16}$ of the beam will get through. See the next page for a picture.

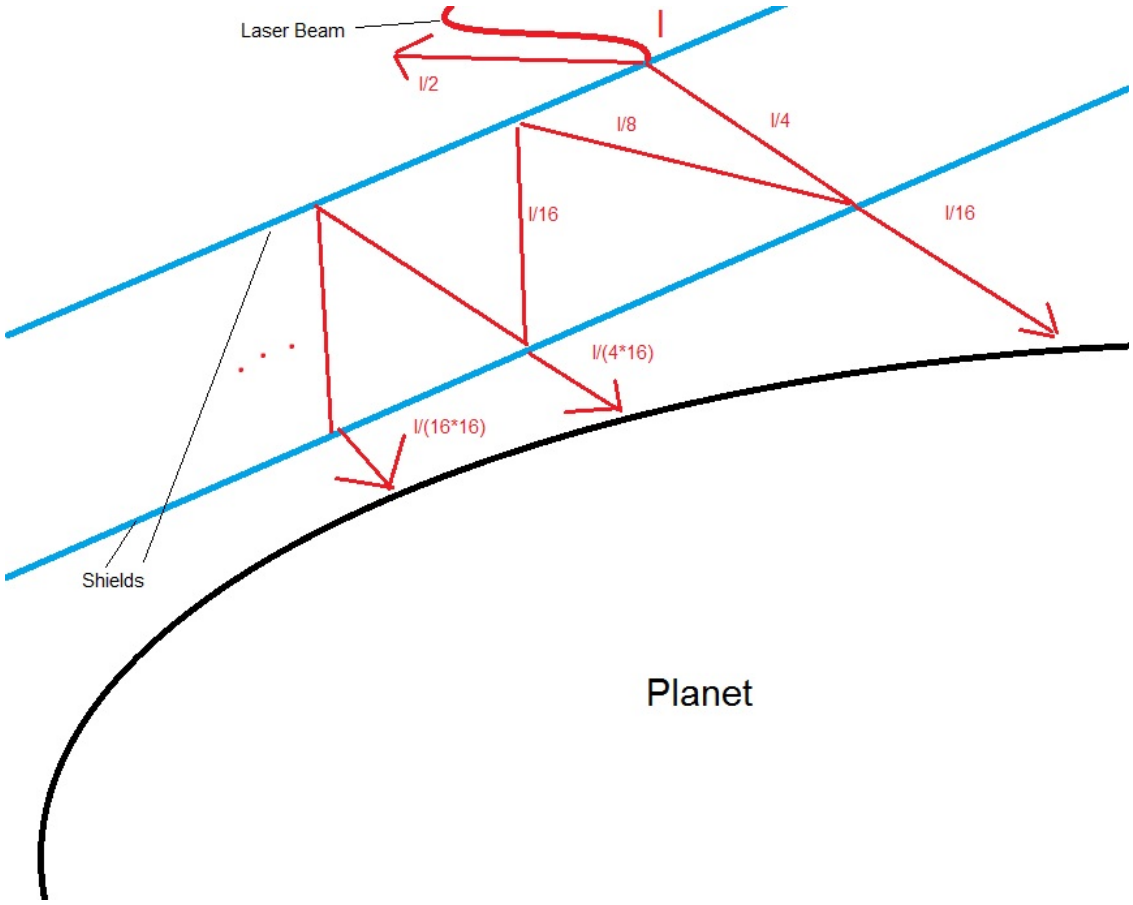
The total amount that gets through is

$$\frac{I}{16} + \frac{I}{4(16)} + \frac{I}{4^2(16)} + \cdots$$

which sums to

$$\frac{I}{16}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{I}{16}\cdot\frac{1}{1-\frac{1}{4}} = \frac{I}{12}.$$

One twelfth, not bad! In the exercises we do a bit better though.


Laser Beam
I
I/2
I/8
I/4
I/16
I/16
I/(4*16)
I/(16*16)
Shields
Planet

5 Closing Remarks

Even though we can easily find the sums of geometric series it's not always so easy for other series. For example, the series $\sum_{n=0}^{\infty} \frac{1}{n^2}$ can be shown to converge. However, it's sum is not so easily found, it turns out that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Another example is

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

and we do not know a closed form expression for $\sum_{n=1}^{\infty} \frac{1}{n^3}$! The fact that we can find the sum easily for geometric series makes them extra special in practice.