MATH CIRCLES FEBRUARY 25 - NOTES

IAN PAYNE

1. A basic subtraction game

The first game we will look at goes as follows: There is a pile of stones. The players, Cathy and Earl, take turns removing either one or two stones from the pile. The player that takes the last stone wins. Cathy goes first.

Example 1. Start with a pile of 15 stones.

- Cathy removes 1, leaving 14.
- Earl removes 2, leaving 12.
- Cathy removes 2, leaving 10.
- Earl removes 2, leaving 8.
- Cathy removes 2, leaving 6.
- Earl removes 1, leaving 5.
- Cathy removes 2, leaving 3.
- Earl removes 1, leaving 2.
- Cathy removes 2, and wins!

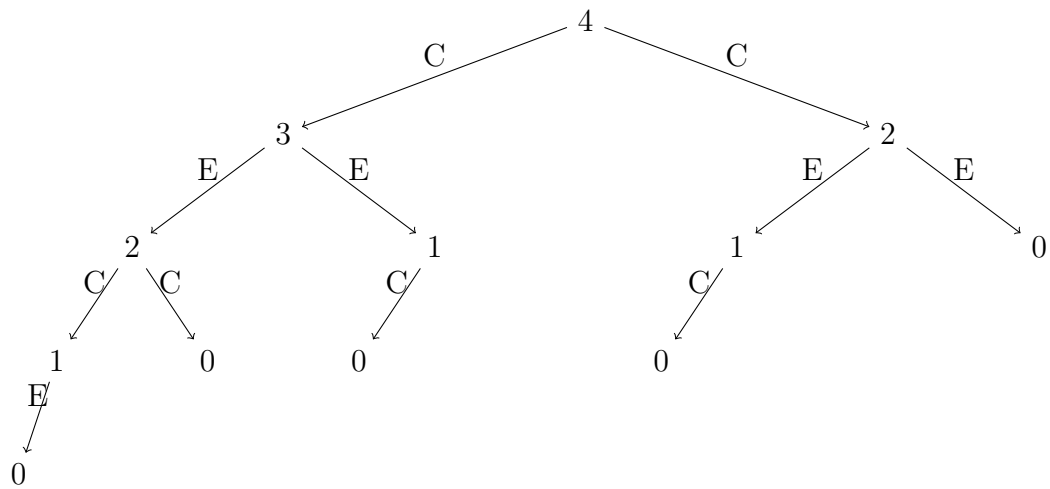
This is what's known as a combinatorial game. Each player is allowed the same moves, and both players can see everything. Informally, you might say that there is no luck in this game. Since both players can see everything, is it possible that one of the players can guarantee a win? The answer is yes, and in Example 1, it should have been Earl. Let's analyze this game more carefully, gradually working up from a smaller number of stones. By the end of this discussion, you should be able to tell who should win the game and how depending on how many stones there are at the start.

- Although it is kind of silly, if there are zero stones at the beginning, we will artificially declare that Earl wins. This makes sense if we rephrase the game ending criterion as "the first player that cannot remove a stone loses".
- If there are either 1 or 2 stones at the start, Cathy can win immediately.
- If there are 3 stones at the beginning, Cathy is cooked. If she removes 1, Earl will remove 2 and win, and if she removes 2, Earl will remove 1 and win.

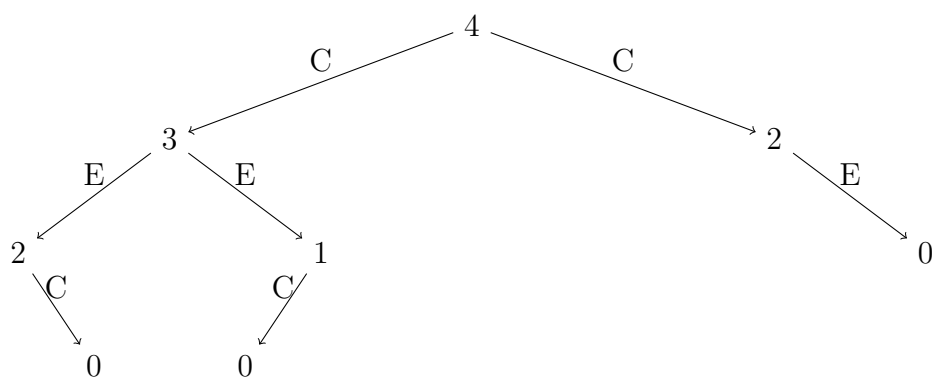
So far, we have

# of stones	0	1	2	3
winner	E	C	C	E

and it generalizes pretty tidily. Let's look starting with 4, and chart the possible moves. An arrow labelled by an initial indicates that person changed the number of stones from the initial number to the final number.



The flow chart above simply maps out all possible moves. Obviously, a player should win if they can, so let's remove any moves that are alternative to a move that will remove all of the stones.



If Cathy removes two stones for her first move, Earl can win. However both of the branches on the other side end with a win for Cathy. So Cathy wants to leave Earl with 3 stones. Think of it this way: If Cathy leaves Earl with 3 stones, then it is as if it is a new game where Earl goes first and starts with 3 stones. We know from previous analysis that the first player loses if there are three. By the same reasoning, if the game starts with 5 stones, Cathy can remove 2 and leave Earl with 3. However, if the game starts with 6 stones, no matter what Cathy does, Earl can stick her with 3 on her next move. Now we have the following:

So far, we have

# of stones	0	1	2	3	4	5	6
winner	E	C	C	E	C	C	E

and you may see a pattern starting to develop. Since starting with 6 is bad, sticking your opponent with 6 should be good. This means that if the game starts with 7 or 8 stones, Cathy can win by leaving Earl with 6, and then following the strategy for 4 or 5 stones depending on what he does. In general, if Cathy can ever leave Earl with a multiple of 3, she can leave him with the next multiple of 3 by making the opposite move to Earl's. That is, if Earl removes 1 stone, she should remove 2, and if he removes 2, she should remove 1. This will leave him with all the multiples of 3 until he has 3 left. Once he has 3, he's done for. On the other hand, if the game starts with a multiple of 3 stones, Cathy is out of luck since Earl can always leave her with multiples of 3. Here is the example with 15 stones, played correctly.

Example 2.

- Cathy removes 1, leaving 14.
- Earl removes 2, leaving 12.
- Cathy removes 2, leaving 10.
- Earl removes 1, leaving 9.
- Cathy removes 2, leaving 7.
- Earl removes 1, leaving 6.
- Cathy removes 2, leaving 4.
- Earl removes 1, leaving 3.
- Cathy removes 2, leaving 1.
- Earl removes 1 and wins.

Forgetting about Cathy and Earl, we have shown that if the number of stones in the pile at the start is a multiple of 3, the second player should win. Otherwise, the first player should win. The strategy is to leave your opponent with a multiple of 3. The first player to do this can continue to do it by always following with the opposite move.

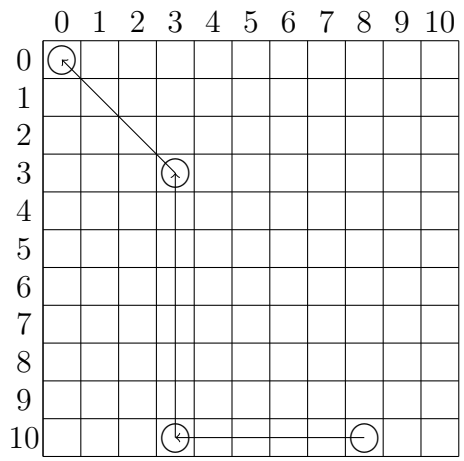
A slight variation on this game is that players are allowed to remove 1,2, or 3 stones from the pile. The key here becomes leaving your opponent with multiples of 4. If you leave your opponent with a multiple of 4, say $4n$, and they remove k, then you can remove $4-k$ since if k is 1,2, or 3, so is $4-k$. This means that they are left with $4n-k-(4-k) = 4n-4 = 4(n-1)$.

If no mistakes are made, the player that leaves their opponent with a multiple of 4 first can leave them with all of the multiples of 4, all the way down to 0.

Much more generally, you can make the rule that players can remove some number of stones from an arbitrary list of numbers. For example, you might make the rule that players are allowed to remove 1,2,6,7, or 9 stones. We won't go into the strategy here, but it may be fun to think about what happens.

2. Wythoff and the left handed Queen

The next game is simple to explain, but has a slightly more complicated strategy. Imagine a queen somewhere on a Chess board and the players take turns moving her. The player that puts her in the top left corner wins. The catch is that the players are only allowed to move her left, up, or up-left diagonal. The board need not be a standard 8×8 chess board, any rectangular grid will do. Here is an example of a sequence of legal moves.



In the example, the first player wins. Can we figure out who has a winning strategy in general?

Let's focus on the 11×11 grid and label the grid with coordinates as in the example (vertical,horizontal). This means that the starting position in the example is $(10, 8)$. Let's label the cells with F if the first player has a winning strategy if the Queen starts there, and S if the second player has a winning strategy. We will adopt the convention that if the Queen starts in the top left position, the second player wins. It's not hard to see that if the Queen starts in the top row, left column, or on the diagonal (other than the top left cell) the first player has a winning strategy. This is because they can move the Queen to the top left corner and win right away. With this observation, we can partially fill out the grid as follows:

	0	1	2	3	4	5	6	7	8	9	10
0	S	F	F	F	F	F	F	F	F	F	F
1	F	F									
2	F		F								
3	F			F							
4	F				F						
5	F					F					
6	F						F				
7	F							F			
8	F								F		
9	F									F	
10	F										F

If a player ever moves the Queen to a cell labelled by an F, they give their opponent a winning strategy. This is because from the opponent's perspective, after this move THEY are the first player to move and the Queen is in a cell labelled by an F. In the cells $(1, 2)$ and $(2, 1)$, there is no choice but to move to a cell labelled by F. This means that if the Queen starts in one of these two positions, the second player wins. For this reason, we label these cells with S:

	0	1	2	3	4	5	6	7	8	9	10
0	S	F	F	F	F	F	F	F	F	F	F
1	F	F	S								
2	F	S	F								
3	F			F							
4	F				F						
5	F					F					
6	F						F				
7	F							F			
8	F								F		
9	F									F	
10	F										F

Now if you think about it, if you move the Queen to a cell labelled by S, you have a winning strategy. Similar to the reasoning from before, from the perspective of the opponent, you become the second player and the Queen starts in a cell labelled by S. Therefore, any cell with "access" to a cell labelled by S should be labelled by F. That is, if the Queen is in a cell with a legal move to a cell labelled by S, it is a first player win. This means we can continue to partially label the grid:

	0	1	2	3	4	5	6	7	8	9	10
0	S	F	F	F	F	F	F	F	F	F	F
1	F	F	S	F	F	F	F	F	F	F	F
2	F	S	F	F	F	F	F	F	F	F	F
3	F	F	F	F	F						
4	F	F	F	F	F	F					
5	F	F	F		F	F	F				
6	F	F	F			F	F	F			
7	F	F	F				F	F	F		
8	F	F	F					F	F	F	
9	F	F	F						F	F	F
10	F	F	F							F	F

Continuing the same reasoning, a cell will be labelled by S if the only legal moves from it are to cells labelled by F. Conversely, a cell is labelled by F if there is ANY move from it to a cell labelled by S. To find the next cells that get S labels, we need to think about what cells are already labelled. If you look at the grid for long enough, you'll see that the so far labelled cells are exactly those pairs (m, n) such that at least one of the following holds:

(1) m is $0, 1$, or 2,
(2) n is $0, 1$, or 2,
(3) $|m - n|$ is 0 or 1.

You should convince yourself of this before you read on.

Of those labelled cells, the only ones labelled by S are $(0, 0)$, $(1, 2)$, and $(2, 1)$. Our goal now is to find new cells that only have legal moves to F-cells. From the picture, we might guess $(3, 5)$ and $(5, 3)$. Indeed, you can see from the picture that the only legal moves from either of these cells are to F-cells. Let's analyze this situation more carefully so that we can generalize this observation. If you think about it, moving vertically is the same as reducing the first coordinate, moving horizontally is the same as reducing the second coordinate, and moving diagonally is the same as reducing both coordinates by the same amount. Note that in this third one, the difference between the two coordinates does not change. For example, if we move $(3, 5)$ diagonally two cells, it changes to $(1, 3)$ and $|3 - 5| = |1 - 3| = 2$. First of all, if we move up or left, we either leave the first coordinate alone, or leave the second coordinate alone. Either way, one of the two coordinates of the resulting pair will be bigger than 3 (think about this). Since none of the so far labelled cells have a coordinate bigger than 2, moving up or left can not send the Queen to an S-cell. Also, since the difference between 3 and 5 is 2, the result of moving diagonally will also have coordinates that differ by 2. None of the so far labelled S-cells have this property, so we can not legally move the

Queen from $(3,5)$ or $(5,3)$ to an S-cell. You should convince yourself that any legal move from either of these cells lands on a labelled cell (a cell satisfying (1), (2), or (3), above). We now conclude that every legal move from $(3,5)$ or $(5,3)$ is to an F-cell. We can now safely label these cells by S. Of course, this now means that any cell with access to either of these cells must be labelled by F. After doing this, we will have labelled every cell satisfying (1), (2), or (3) above, as well as those satisfying one of

(4) m is 3 or 5,
(5) n is 3 or 5,
(6) $|m-n| = 2$.

We now leap to a general way to construct all cells labelled by S. Let $p_n = (a_n, b_n)$ be the nth S-cell. Here we note the symmetry by only recording the pair with the left coordinate less than the right. That is $p_0 = (0,0)$, $p_1 = (1,2)$, $p_2 = (3,5)$ and we ignore $(2,1)$ and $(5,3)$. Notice that so far, if $p_n = (a_n, b_n)$ that $b_n - a_n = n$. We started with $a_0 = (0,0)$ and constructed the subsequent pairs recursively. The next pair needed to have the following properties:

- Each entry is bigger than any entry in any previous pair. Ootherwise, a vertical or horizontal move can take the Queen to one of the identified S-cells.
- The difference between the two entries is larger than the difference between the coordinates of any previous S-cell. Otherwise, a diagonal move could take the Queen to an S-cell.

In more mathematical terms: Given that we know $p_0, p_1, \ldots, p_n$, to find a_{n+1}, pick the smallest positive integer that is greater than all of $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$, and then to find b_{n+1}, just take $b_{n+1} = a_{n+1} + (n+1)$ to ensure their difference is big enough. For example, since $a_0 = 0$ and $b_0 = 0$, the smallest integer greater than both of these is 1 so $a_1 = 1$, and $b_1 = a_1 + 1 = 1 + 1 = 2$. The smallest integer bigger than a_0, a_1, b_0, and b_1 is 3, and $3 + 2 = 5$. This means $p_2 = (3,5)$. Continuing in this way, we get the following table:

n	a_n	b_n
0	0	0
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26

The winning strategy in this game is to move to one of these pairs (or the opposite pair). Unless you are already on one of them, there is always a way to get to one. Try it!

Now for another game. It's called Wythoff's game. There are two piles of stones, two players that alternate moves, and the player that takes the last stone wins. The legal moves are to remove stones from one pile, or the same number of stones from both piles. If you think about it, this game is really the same game as the Chess board game.