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Grade 6 Math Circles

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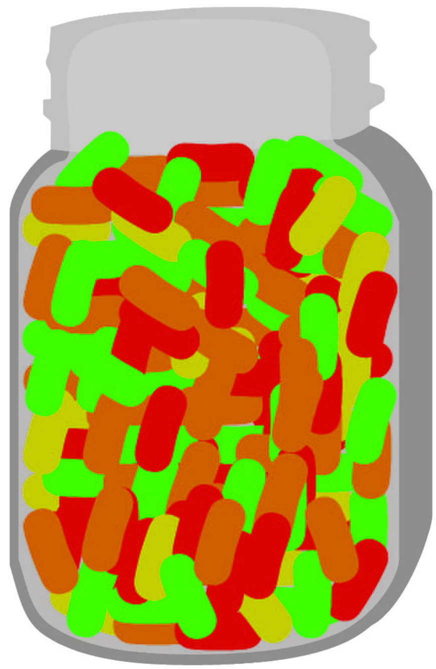
Estimation

The Candy Jar Contest

You have probably heard of candy jar contests before. Your goal is to guess a number as close as possible to the true number of candies in the jar. How would you make your guess? Is there a strategy you could use to make a better guess than other contenders? A mason jar's sides are close to being rectangles. You are allowed to observe the jar but you cannot measure it with a tool. Believe it or not, there is a way to guess the number of candies and come reasonably close. In fact, I won three contests like these in secondary school because I was the only person to use a strategy



What do you know about the container of candy? It is a mason jar and resembles a rectangular prism. You also know how to calculate the volume of a rectangular prism: multiply the length times width times height. Can you figure out how to use these facts to guess the number of candies? How do you get around not being able to use a ruler? Do you think you could make a good *estimate* if you only had two minutes? I am claiming it is possible and that I did it successfully in three out of four contests I entered in school. We need to learn how to make a good estimate.

Take Your Best Guesstimate!

Guesstimates are estimates of a value without all of the information you need for a more accurate estimate. Later, when we guess the amount of candy in the jar, we will use a strategy to make an estimate. Someone who does not have a strategy for estimating the candy, but writes down a number anyway, is making a guess. So a guesstimate involves a partial strategy that may (or may not!) make the guess more accurate.

Guesstimate (or estimate) the following (compare with the real answer when possible):

The population of Waterloo.	The population of Ontario.
The distance to Toronto.	The length of your feet in cm.
The distance to New York.	The length of the room in m.
The time to walk to the bathroom.	The time that ice-cream takes to melt.
How much energy to kick with in soccer.	The weight of a grocery bag.

Why did I have you guess these measurements? The purpose of guesstimates is to obtain an answer that is not unreasonable. This means that it is accurate enough to use for a simple purpose. Having an intuition for guesstimates can be extremely useful and even help when you want to estimate. If I asked you to draw a line that is an inch long, how close would you be to drawing a true inch? You may guess the length right now but if you practice you may actually improve your guesstimate into an estimate. One should be careful, although you can become quite good at guesstimating, you should not use guesstimates when accuracy matters. In a lot of mathematics, accuracy is very important, or if you become a doctor and must measure medicine in the future, accuracy is extremely important.

Other disciplines require estimates and some only need guesstimates. Physicists and economists use guesstimates in specific situations when more information is not available but they require some basic solution. To guess the number of candies, we should first learn how to estimate measurements properly. Grab a piece of paper and guess how long $1cm$ is *and* $10cm$ are by drawing lines of that length.

Here are the two lengths you just guessed.

$1cm$ _____ $10cm$ ____

There are a few techniques to estimating these lengths without a ruler. I estimate these measurements with my thumb, index finger, and eyes. I move my thumb and index fingers apart until the space between them is slightly larger than $10cm$ (I think) and then I go back and forth between making the space slowly smaller and larger until I have what I think is a measurement close to $10cm$. I do the same for $1cm$; I may even try estimating $10cm$ by sizing about 10 of my 1 centimetre lengths.

Another, perhaps more accurate technique, is to use your fingers and hands as estimators. Measure your hand with a ruler to find a width of a finger, the length of a finger, or some other feature that is about the length of a centimetre, ten centimetres, and an inch. For example, the width of my pinky is about a centimetre and the tip of my index finger to the first joint is about an inch in length. This allows me to quickly measure small items. If I want to measure longer distances, my shoe is about a foot in length and a metre is about the same distance from my heart to tip of my outstretched right arm and hand. You will likely need to find different ways to measure these lengths since we are probably not the same size.

Think about why these are estimates rather than guesses or guesstimates.

Exercise: Use the rest of this space to practice drawing lengths between 1 and 10cm. You should at least estimate 2,5,6, and 9cm lengths. You may not use a ruler but you may use the lines at the top of the page for reference.

Estimating Other Quantities

Everyone needs to make guesses on a daily basis. It is a valuable life skill to be able to estimate and guesstimate. If you are using cash to buy something, estimating tax is useful. If you need to be somewhere by a certain time, estimating how long it will take for you to get there is also useful. Maybe you are buying something online and want to see how heavy it will be but cannot feel the weight, then you should estimate the weight.

Let us see how to estimate tax. In Ontario, there is a 13% Harmonized Sales Tax on most products. I use a mathematical rule called the distributive law to estimate 13% of my subtotal. Here is how it works:

Say we have two numbers being added together, for example 3 and 4. After we add them, we want to multiply them by some number, say 5. I would write this mathematically as $5 \times (3 + 4)$ with the brackets to mean I should add first. But the distributive law states that multiplication can be distributed over addition (or subtraction). What this means is that $5 \times (3 + 4)$ is actually the same as (equal to) $5 \times 4 + 5 \times 3$. See how the 5 has been *distributed* out to the other numbers? Try it out for yourself on the following examples; it always works!

$3 \times (3 + 3) = 3 \times 3 + 3 \times 3$	$10 \times (4 + 1) = 10 \times 4 + 10 \times 1$	$7 \times (4 - 1) = 7 \times 4 - 7 \times 1$

So how is this useful in calculating 13%?

First, I use the decimal form of 13%, that is, 0.13. Then I notice that $0.10 + 0.03 = 0.13$. Now for whatever subtotal I have, maybe it is \$3, I can use the distributive law:

$$\$3 \times 0.13 = \$3 \times (0.10 + 0.03) = \$3 \times 0.10 + \$3 \times 0.03$$

Now here is where the estimation comes in: I know that 0.03 is not quite one third of 0.10 but it is close. So I can use this fact $0.03 \approx 0.10 \times \frac{1}{3}$ to make calculating $\$3 \times 0.03$ a little easier. That is, I can think of it as:

$$\$3 \times 0.03 \approx \$3 \times 0.10 \times \frac{1}{3} = (\$3 \times 0.10) \times \frac{1}{3}$$

where I calculate what is in brackets first. But the bracketed expression I have to calculate anyway! So I have saved us some work. And multiplying by 0.10 is easy, it is the same as dividing by ten, or moving the decimal place one place value to the left. Then I just need to estimate one third of that amount and add it on as well. So we have:

$$\$3 \times 0.13 = \$3 \times 0.10 + \$3 \times 0.03 \approx \$3 \times 0.10 + (\$3 \times 0.10) \times \frac{1}{3}$$

So 3 divided by 10 is 0.30 and then one third of 0.30 is 0.10 so we have:

$$\$3 \times 0.13 \approx \$3 \times 0.10 + (\$3 \times 0.10) \times \frac{1}{3} = 0.30 + 0.30 \times \frac{1}{3} = 0.30 + 0.10 = \$0.40$$

So my tax will be about 40 cents!

If you do not like multiplying by one third, or dividing by three, then you could estimate 13% by 15% and use the fact that $0.15 = 0.10 + 0.05$ and 0.05 is exactly half of 0.10 so where we divided by 3 you could instead divide by 2.

Statistics

If you want more accurate estimates, sometimes you can collect data on what you are estimating. For example, I could record how long it takes me to get ready in the morning for this whole week. This would give me a good idea of how long it will take me to get ready next week. What I am doing is collecting a *sample*. In statistics, a sample is a bunch of data points (for example, a list of times in minutes to get ready). A sample is typically a small selection of a larger population and in our example we have picked a very small sample. Could this affect how good of an estimate we gather? Yes. What if this week I work and next week I take a vacation? The time it takes me to get ready on vacation might be different from the times before work. How do we work around this variability?

Statisticians have a few practices to prevent this *bias* from happening. A *bias* is a factor that can occur at any point while collecting or analysing statistics that introduces some error. In our example, collecting a larger sample would remove some of the bias. I could measure my time in the mornings for a whole year and that may give me a better estimation for the following weeks. However, I may not always remember to record my time throughout a year so instead we could use randomness. I could randomly select seven days out of a month as my sample. This hopefully mean that the days I have selected are normal times and not be biased by abnormal activities. Even better, we could use both techniques when collecting a sample.

We could apply this method to estimating the number of candies in our mason jar. After filling mason jars identical to ours with candy, we could write down how much fit in each. We now will calculate the average or *mean* number of candies from all of the jars. The mean is an estimate of the candy in a typical jar and if given any mason jar filled with candy I could expect it to contain close to the mean number. To calculate the mean we add all of our data points and divide by how many there are. For example, if I sample 5 jars that contain 345, 356, 323, 355, 340 candies each their average is:

$$(345 + 356 + 323 + 355 + 340) \div 5 = 1719 \div 5 = 343.8 \approx 344 \text{ candies}$$

So my guess would be 344 candies. If we be careful of biases and other errors, our estimate will be more accurate.

Including the mean, here are three popular measurements in statistics:

- Mean: Add all of your values together and then divide by how many there were.
- Median: The middle number of an sorted list (ex. $1, 5, 5, 9, 22$ has median 5).
- Mode: The number(s) that occur most frequently in the data (5, in the last example).

A mean is easy to calculate. A mode is somewhat easy to find if you order your data from least to greatest. A median can be a little more difficult. The trick to finding a median is first ordering you data and then counting how many numbers you have. Take that number of numbers, n, and add 1, then divide by 2. This DOES NOT calculate the median but instead gives the location of where the median will be. Count the numbers and find the number in that position, it is your median.

For example: $2, 3, 3$. Obviously the middle number is 3 so it is our median. Following our rule, if we take the number of numbers 3 add 1 and then divide by 2: $(3+1) \div 2 = 4 \div 2 = 2$ then that gives the position 2. 3 is in the second position of our list and is our median, as we expected.

With an even number of numbers, such as $1, 3, 4, 4$, we follow our trick and find that the position is $(4 + 1) \div 2 = 5 \div 2 = 2.5$. But there is no 2.5th position. Well 2.5 is exactly halfway between 2 and 3 so we instead calculate the mean of the 2nd and 3rd numbers. The 2nd number is 3 and the third is 4. Their mean is: $(3 + 4) \div 2 = 7 \div 2 = 3.5$. So this data has median 3.5.

Estimating the Number of Candies

You may have already determined how to do this. The key here is to estimate the volume of the jar and the volume of one candy. Take the volume of the jar and divide it by the volume of each candy and you will get an estimate of the number of candies. The mathematics is not difficult, you just need to be a good estimator.

As you have seen, there are numerous other ways of estimating this value but this is perhaps the quickest and one, as I have claimed, you can do entirely with your fingers and still have a better chance to win in a guessing contest.

Exercises

1. Estimates can be useful when making quick calculations. Try rounding the numbers in the following parts to the nearest hundred, ten, or one (your choice) and then determine the answer in your head. Note: You should round so that it is easier. Calculate the exact answer; how close was your estimate?

$$\begin{array}{lll} 3 \times 17 & 22 \div 5 & 24 \times 12 \\ 347 \times 21 & 678 \div 5 & 1345 + 894 - 201 \end{array}$$

2. Samir measures the mason jar with his thumb. He says it is 9 thumbs tall, 5 thumbs wide, and 5 thumbs deep. He also says that the candies are one thumb tall, one thumb wide, and one thumb deep.

   (a) Using your thumb, make an estimate of how tall, wide, and deep the jar is in centimetres.

   (b) What is the approximate volume in cm^3 with these measurements?

   (c) Why is this an estimate and not a more *exact* measurement like one obtained with ruler?

   (d) Give an estimate of the number of candies in the jar.

   (e) * Would it be more accurate to convert our units from thumbs to centimetres before estimating the number of candies in the jar?

3. Without using a ruler, draw a 1 cm line, a 10 cm line, and a 7.5 cm line in the space below. After doing so, find a ruler and see how close you were. Have you improved from before?

4. Solve each problem twice. The first time you should multiply first, then add. The second time you should add first, then multiply. Make sure you get the same answer both times. DO NOT USE A CALCULATOR. Hint: Distributive Law.
   $4 \times (8 + 12)$ $\quad$ $32 \times (1 + 2)$ $\quad$ $22 \times (5 + 30 + 100)$

5. Amir is helping Caleb with a guessing game. Caleb must guess the volume of an item and to do so he is given the following information:
   - The item has the shape of two cubes glued together, a smaller one on top.
   - The width of the large cube is 15cm.

   (a) Using this information, guess, guesstimate, or estimate the volume of the item.

   (b) Which did you use: guess, guesstimate, or estimate? What is the best you can achieve with the information you are given?

   (c) Amir reads the rules again and finds another piece of information to use: the item is about as tall as a common banana.

      i. Guess, guesstimate, or estimate the volume of the item using the new info.

      ii. Which did you use: guess, guesstimate, or estimate? What is the best you can achieve with the new information you are given?

6. Calculate the mean for each of the two sets of data. If we were to estimate the size of the next number, do you think the mean is a good estimate of the size? For example, if 5 is the mean of $5, 5, 5, 5, 5$ then if we collected one more piece of data, would the mean, 5, be a good guess of our new datum?

   (a) $3, 4, 1, 5, 2, 5, 2, 6, 1, 3, 4, 1, 1, 1, 3, 5, 2, 4, 6, 3, 2, 3$

   (b) $5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 6, 70, 100, 5, 5, 5, 5, 5$

7. Match the following options to your best guesstimate for their size, be as accurate as you can be.

   (a) The height of a house cat.

   (b) The length of a golden retriever.

   (c) How many years ago was your great great great great grandmother born?

   (d) How long ago did ancient Rome fall?

   (e) The circumference of the earth.

   (f) The distance from Waterloo to Tokyo, Japan.

   (g) The travel time from Waterloo to Toronto.

   (h) The travel time from London to Toronto.

   (i) The weight of a cast-iron bathtub.

   (j) The weight of a marble statue.

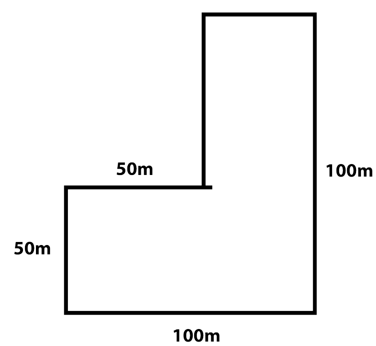
   (k) How many times does a person sneezes in their life?

   (l) How many chickens are there in the world?

   (m) How many species of beetle are there?

- 200 years
- 2 hours 20 minutes
- 10 000 km
- 350 thousand
- 1 hour 30 minutes
- 30 centimetres
- 15 thousand
- 83 centimetres
- 5,660 kg
- 50 billion
- 454 kg
- 1500 years
- 40,075 km

8. Your parents are worried about finding parking for the Math Circles. They worry that there will not be enough space available. You know that there are four Math Circles events, one for each of Grade 6, 7/8, 9/10, and 11/12. Two sections contain about 35 students each and the other two sections contain about 70 students each. You guess that just about everyone arrives in different cars – even if you are incorrect in this assumption, you know the parking lot needs to be able to hold that many anyway. Assume a vehicle is about 2m wide and 3m long. Use the image below to estimate how many cars will fit in the parking lot. Will your parents find a spot?



9. Brien is a programming a computer to win a game. He wants to design an algorithm to guess a number from 1 to 300 that you pick. To win the game, Brien must guess the number in under 10 tries. Every time the algorithm guesses a number, you must tell it if its guess was too high or too low. You cannot lie about your number.

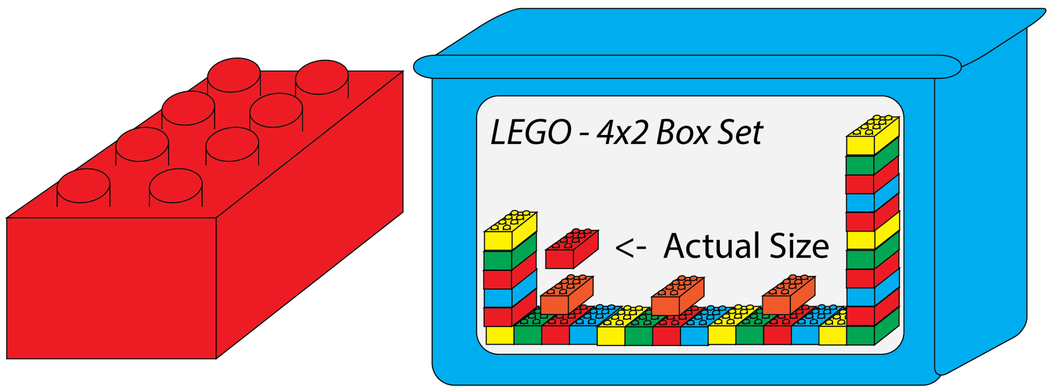
   (a) Brien's first algorithm guesses the numbers from 1 to 300 until its guess is correct. Will this algorithm win the game? Can you think of a better way to guess?

   (b) Brien's second algorithm is called a binary search algorithm. Find a partner and play the guessing game with them. Use Brien's algorithm:

   - Always guess the median of the remaining set of numbers.

   Does Brien's new algorithm win the game for you? * Will it always win the game?

10. * You work at a toy store which has just received a shipment of Lego bricks. The bricks came in boxes but the Lego company forgot to put how many bricks are contained in each box on the front. Each box contains all the same type of block, a 4×2 piece as pictured below. You cannot open the box to see how full it is but you do have a 4×2 block and the boxes for reference. Describe the process you would use to estimate the number of blocks.



11. Jessica and Sam are giving their class a survey. They want to determine the average height of Grade 6 students in Canada so their only question is "How tall are you in centimetres?" Their class has 28 students and the results are listed in the following table:

156	146	135	155	164	134	144
150	149	160	158	145	145	143
142	147	148	150	151	143	153
139	141	140	149	145	152	152

Give an estimate for the average age of a Grade 6 student in Canada.

* Would you say that Jessica and Sam's estimate is accurate?

12. * How would you prevent someone from estimating how much candy is in a mason jar?