Math Circles - Surfaces, Selected Solutions

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Question Sheet 3

7. This question has been steadily getting easier, with the tools we've been developing, and now with the word manipulations, it has become the easiest it will ever get! Given any word W, if we take its connected sum with a sphere, the new word is Waa^{-1}, which, by rule 6 is equivalent to W. Thus, taking a connected sum with a sphere does nothing, and the resulting surface is equivalent to our original one. This is what our gut told us initially. If you take a connected sum with a sphere, you should be able to just push the sphere in on itself, having it completely disappear back into the original surface (kind of like releasing the air from a balloon).

8. Since all orientable surfaces have even Euler characteristic, we know this surface is non orientable. Using $\chi(A\#B) = \chi(A) + \chi(B) - 2$, we see

$$\chi\left(\overbrace{PP\#\cdots\#PP}^{n\text{-times}}\right) = 2 - n.$$

   Therefore, this particular surface is equivalent to $PP\#PP\#PP\#PP\#PP\#PP\#PP$.

9. We know this is a non-orientable surface, and using $\chi(A\#B) = \chi(A) + \chi(B) - 2$ we know it has Euler characteristic -5. Therefore the answer is the same as in question 8: the connected sum of 7 projective planes.

10. A 3-dimensional surface (or to use the words the world of mathematics uses, a 3-manifold) would be something which when you zoom in close enough and look around, it just looks like a piece of 3-dimensional plane. Although (next to) impossible to visualise, a 3-dimensional torus would be something with a 'planar model' given as follows. Take a room and if I walk through one wall, I would end up at the opposite wall, if I walk through the ceiling (gravity is a minor technicality here) I would come up through the floor! Kind of like what a 3-dimensional pacman game would be!

    The Euler characteristic would be calculated in much the same way, except we also have number of "blocks" in the equation now. We can define the Euler characteristic of a 3-manifold X as

$$\chi(X) = V - E + F - B$$

    where B is the number of 3-dimensional blocks needed to make up the surface. For example, if we take the model described above of the 3-dimensional torus it has 1 vertex (they're all glued together), 3 edges, 3 faces and 1 block. Thus χ(3-dimensional torus) = 0. In fact, something weird happens in 3-dimensional surfaces. All the Euler characteristics (well if you put some restrictions on your surfaces) are 0, so as an invariant, it is useless to us!

Unfortunately, we don't have as nice a classification theorem for 3-manifolds as we do for 2-manifolds. However, we do have quite a bit of progress here in the form of the Geometrisation Conjecture, which was conjectured by Bill Thurston in 1982 and proven by Grigori Perelmen in 2003. This is one of the tools used in Perelmen's proof of the Poincaré conjecture, the only Clay millenium problem (which were 7 problems set in 2000 each worth 1 million dollars) to be solved today.