Math Circles - Surfaces

Night 1

Tyrone Ghaswala - ty.ghaswala@gmail.com

3rd February 2016

Introduction

This instalment of math circles is intended to give you a taste of an area of pure mathematics that you (hopefully) haven't seen or even had a sniff of before. The layout of the course will be as follows. We will first create a universe with its laws and explore a few examples of objects that might exist in there. This universe, the universe of 2-dimensional surfaces, will seem extremely poorly lit at first and we will feel like we are walking around in the dark. So we will do the only thing we can do, walk around and see what happens.

Along our explorations we will bump into some surprising (sometimes terrifying) things and occasionally we will stumble upon a light switch, which will clear things up at least until that next dark corner. With a bit of luck, by the end of the three sessions our beautiful universe will be completely lit up in all its glory for us to see.

Because this area of mathematics is intended to be brand new to just about all of you, you do not need any background in just about anything to follow what is going on here. All you will need is your imagination and a willingness to explore.

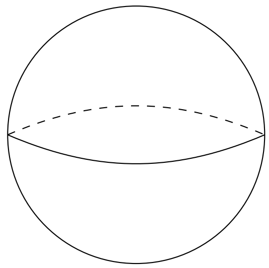
Before we begin it should be noted that this topic was first introduced to me by Ben Burton (who's currently at the University of Queensland in Australia) and the way I will be presenting it is almost entirely inspired by the way he taught it to me.

Let's go!

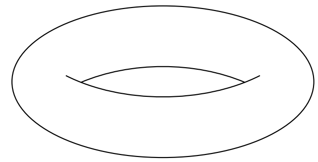
The Universe of 2-Dimensional Surfaces

Let's create the universe we are going to be exploring. Our universe that we will interested in is the universe of 2-dimensional surfaces. So, what kind of objects exist in our universe and how are we allowed to play with them? What are the "laws of physics" in this universe?

Two examples of such objects are the surface of a **sphere** and the surface of a donut (a **torus**), but we only care about the surface! You should picture these in your head as being hollow, so in some sense they are 2-dimensional things (since the surface is 2-dimensional, I will explain this a little more later on).



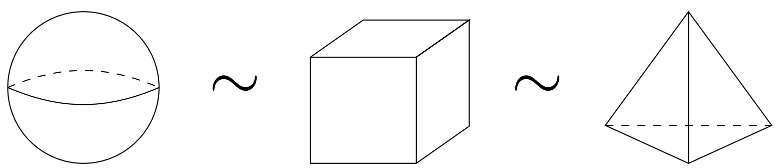
Sphere



Torus

So, what are we allowed to do with these objects? It turns out that the objects in our universe are made of spandex (or at least you can think of them that way). That is, we are allowed to stretch, shrink, squish, and deform our surfaces in any way we like. However, we **CANNOT** cut or glue our surfaces.

For example, the sphere (in this universe) can be legally deformed into a cube, or a tetrahedron since we can just squish the sphere until it looks like one of these without cutting or gluing. We can also stretch it out into a long snake, and then return it to a sphere.



Some legal deformations of the sphere

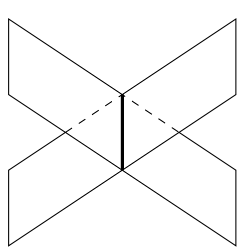
The torus can be legally deformed into a coffee cup (remember, we're only considering the surface of it) and back. The hole in the torus becomes the hole in the handle of the coffee cup.

So, now that we have some examples and maybe a bit of instinct as to what's allowed and what is not, let's define this universe a little more concretely.

Definition. A **2-dmensional surface** is an object that, when you get sufficiently close, just looks like a bent piece of a 2-dimensional plane.

This idea is something we're very familiar with on the surface of the Earth. If we get sufficiently close to the surface of the Earth (say about 180 centimetres or so) and look around, the surface of the Earth looks a lot like a bit of 2-dimensional plane (in fact so much so that the human race used to think it *was* a plane, and not a sphere!).

An example of something which is *not* a 2-dimensional surface is the object pictured below. If you zoom in to any point other than the intersection and look around, it sure looks like a piece of 2-dimensional plane. However, if we zoom in to any point on the intersection and look around, it always looks like 2 planes intersecting, it doesn't matter how close we get.



Not a 2-dimensional surface

Definition. Two surfaces, call them S and T, are **equivalent**, and we write $S \sim T$, if we can get one from the other.

So, we know the sphere is equivalent to the cube, and the torus is equivalent to a coffee cup! Now, before we get down to exploring this universe there are a couple of things worth noting.

- We are only concerned with surfaces that are not infinite and do not have boundary.
- We are not confined to the surfaces living in 3-dimensions.

What do these two points mean exactly? Well, not being infinite means we don't go off infinitely far in any direction. Not having a boundary means that I can walk along the surface and never reach an edge (which I cannot do on a piece of paper for example, which means a piece of paper has a boundary). Not being confined to 3-dimensions means that we are allowed to have two 'bits' of a surface pass through each other. Even though this might seem like we are cutting our surface to allow it to pass through itself, this is just a 3-dimensional representation of what is going on. If we allow more dimensions, then a surface can be legally deformed through a step that appears to be passing through itself, without passing through itself at all (there's enough room to go around instead of passing through). Weird huh!?

Now that we have set up and explored our universe a little (in possibly a more confusing than enlightening manner), let's get down to business!

Big Questions

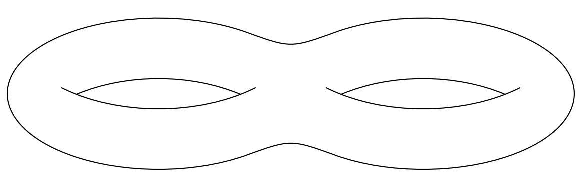
Here are some questions that we will be setting out to answer. These are the big questions we should be keeping in the back of our minds throughout these three weeks.

1. If I give you any two surfaces, can you tell me whether or not they're equivalent? For example, is the torus equivalent to the sphere? We hope they aren't, but how can we prove it?
2. If I give you a surface, can you tell me which surface it is?
3. Can we make a list of all possible surfaces?

With these questions in mind, let's start looking around in our universe!

Some more 2-dimensional surfaces

We have already seen a sphere (which we will call S) and a torus (T). But what other surfaces are there? Well, we can have a 2-holed torus, or a 3-holed torus, or maybe even an n-holed torus! But is that all, and are they even different? Let's keep looking.



A 2-holed torus

Projective Plane

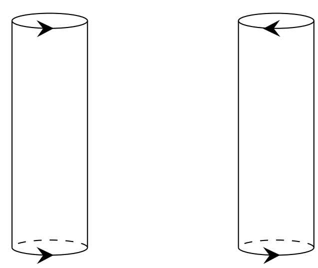
This surface is an interesting one, because we cannot visualise it living in 3-dimensional space (but it does exist in 4-dimensions in our universe!). We will denote it by PP, and here's how we make it.

First, we take a Möbius strip and a disk. We know a Möbius strip has only 1 edge (if you don't believe me, make your own Möbius strip and draw a line along one of the edges and see what happens), and the disk has one edge. Well, let's glue them together! It turns out that this is a 2-dimensional surface and it does live in our universe.

Klein Bottle

Again, like the projective plane, this cannot be visualised in 3-dimensional space without it passing through itself, but it does exist in 4-dimensions! To describe it, we must first take a look at the torus in a little more detail.

One way to think about making a torus is to glue two ends of a cylinder together, but what if you glued the two ends together in opposite directions (as shown in the diagram below, where the arrows mean glue the edges together so that the arrows match up)? Then you would get a Klein bottle, which will be denoted KB.



Torus *Klein Bottle*

New Surfaces from Old

So now we have four examples of surfaces that live in our universe. We still have no idea (although we can guess) whether or not these are equivalent or different, but at least we have something! Now we will explore two apparently different ways of creating new surfaces from old ones.

Adding a Handle

This process is a lot like what it sounds like. Given a surface (call it Bob), we are going to take it from our universe, attach a handle and put it back! Here's how we attach a handle:

1. Cut two holes in Bob. This gives us two edges which are ready to be glued to something.

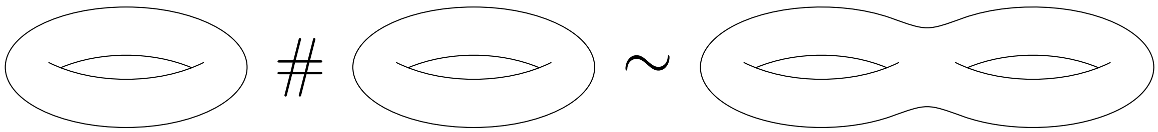
2. Take a cylinder and glue each of the two edges to the two edges on the surface.

Voila! You now have a potentially new surface. See if you can convince yourself that adding a handle to a sphere gives you a torus.

Connected Sum

This is a kind of way of joining two surfaces together. Let's say now we have two surfaces, Jack and Jill, and we want to create a new surface out of these two. Here's what we can do. We can cut a hole in both Jack and Jill, which gives both surfaces an edge, and then glue them both together along that edge! We will call this surface Jack#Jill.

For example, if we take two tori and take their connected sum, with a bit of imagination (try it!) we can see that



$T\#T \sim$ 2-holed torus.

Spotting the Difference

We have a bunch of surfaces now, and we would really like to know whether or not they are equivalent. We guess that the sphere and the torus are not equivalent, but right now we have no way of proving that. If we are to prove it we need some thing, some quantity, number, colour, face, anything that stays the same under legal deformations of the surface. Then if we calculate such a quantity to be, say 7 for the torus and 5 for the sphere, then we know that we can never deform a torus to be a sphere, and we will have proved that they are indeed not equivalent. Such a tool is called an **invariant**. Although this gives us a tool to tell if two things are different, it **does not** tell us when two things are the same. Let's create an invariant now.

Euler Characteristic

We will now learn how to calculate the Euler characteristic of a surface. This will be a number that we can associate to every 2-dimensional surface that is an invariant, and here's how we calculate it.

1. Deform your surface so that it is some sort of polyhedra. That is, deform it until the surface is made up of polygons. For example, we can legally deform the sphere to be a cube, or a tetrahedron.

2. Count the number of vertices (V), edges (E), and faces (F) on your polyhedra.

3. Calculate the quantity $V - E + F$, and this is your **Euler characteristic**. If your surface is A, then to denote the Euler characteristic we write $\chi(A)$.

Let's do an example with the sphere. As we've seen before, the sphere is equivalent to a cube and a cube has 8 vertices, 12 edges and 6 faces. Therefore we have

$$\chi(S) = V - E + F = 8 - 12 + 6 = 2.$$

At this point you should be screaming in protest! This only works if we turn a sphere into a cube, there's no way we will get the same number if we turn the sphere into anything else, right? Amazingly, this is not the case. Let's have a look at what happens if we turn the sphere into a tetrahedron instead. We know a tetrahedron has $V = 4$, $E = 6$, $F = 4$, so we see that $\chi(S) = 4 - 6 + 4 = 2$, which is the same number we got with the cube! Amazing.

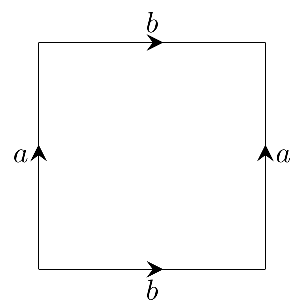
Fact 1. *The Euler characteristic is an invariant of any surface.*

With all of this in mind, we now have the tools to prove that two surfaces are not equivalent. How can we prove a surface is not equivalent to a sphere? Easy, we calculate its Euler characteristic and if it is not 2, then we know the surface is not equivalent to the sphere. Unfortunately, if we find the Euler characteristic of our surface is 2, then we cannot say anything about whether or not it is equivalent to the sphere.

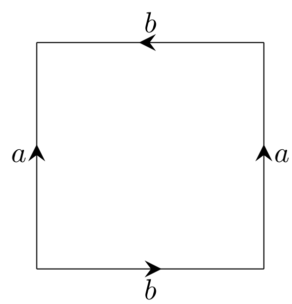
Planar Models

So, now that we have the Euler characteristic, in theory we should be able to start proving some things about surfaces. However, in practice this is not turning out to be the case. The current way of calculating the Euler characteristic of a surface seems prohibitively tedious and difficult, so we need some help. Enter planar models!

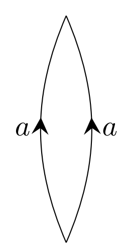
This is a way of visualising surfaces without having to imagine 4 dimensions. We will see it is a useful tool for just about everything! Let's have a look at the planar models of some familiar surfaces (see if you can work these out for yourself, some of them appear in the exercises).



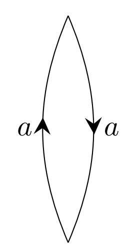
Torus



Klein Bottle



Sphere



Projective Plane