Math Circles - Surfaces

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1. Make a list of all the surfaces you can think of that aren't equivalent. Can you make a list of all of them? How do you know any two on your list aren't equivalent?
2. Calculate the Euler characteristic of the following surfaces.
   (a) The torus.
   (b) The Klein bottle.
   (c) The projective plane.
   (d) A two holed torus.
3. Are the following surfaces equivalent to something we've seen before? If so, what? Can you prove your claims?
   (a) $S\#T$
   (b) $PP\#PP$
   (c) $T\#T$
   (d) $S\#PP$
   (e) $T\#PP$
   (f) $KB\#PP$
4. What happens to the Euler characteristic of a surface S when you add a handle to it?
5. Suppose that S and T are two surfaces. Find a formula for $\chi(S\#T)$ in terms of $\chi(S)$ and $\chi(T)$.
6. Without using the Euler characteristic, prove that the torus is not equivalent to the sphere.
7. **Adding a crosscap to a surface.** Here we are going to define what it means to"add a crosscap" to a surface S. We saw earlier that a Möbius strip has only one edge, so let's make use of this fact. First, cut a disk out of S. This creates an edge which is now available to have something sewed on to it. Sew in the edge of a Möbius strip and voila, you have a new surface which is S with a crosscap added.
   (a) What do you get when you add a crosscap to a sphere?
   (b) What do you get when you add a crosscap to a projective plane?
   (c) What happens to the Euler Characteristic of a surface when you add a crosscap to it?
8. Find a planar model for the Klein bottle and the projective plane.
9. Is there a way you can represent the connected sum using planar models? That is, if you have two planar models of surfaces (call them S and T), can you easily find a planar model for $S\#T$? How about a planar model for adding a crosscap?
10. Make a list of all your conjectures about surfaces and keep them safe. Light will be shed on a lot of things in the next two weeks.