Intermediate Math Circles
Wednesday, April 5, 2017
Analytic Geometry III

What's a Locus?

Definition [Locus]:
A locus is a set of points that satisfy a given condition or the path traced out by a point that moves according to a stated geometric condition.

Examples:

- lines
- circles
- parabolas
- ellipses
- hyperbolas

What is a Ellipse?

- Ellipses are round
- An ellipse is a set of points such that the sum of the distance from each of the two fixed points (foci) is constant.
- An ellipse has a **major axis** (the longer one) and a **minor axis**
- An ellipse has two **focal points (foci)** that are equidistant from the centre on the major axis.
- The two distances from a point on the ellipse to its foci are the **focal radii**.
- The points where the curve crosses the major axis are the **vertices of the ellipse**.

Drawing Ellipses

Ellipse Questions:

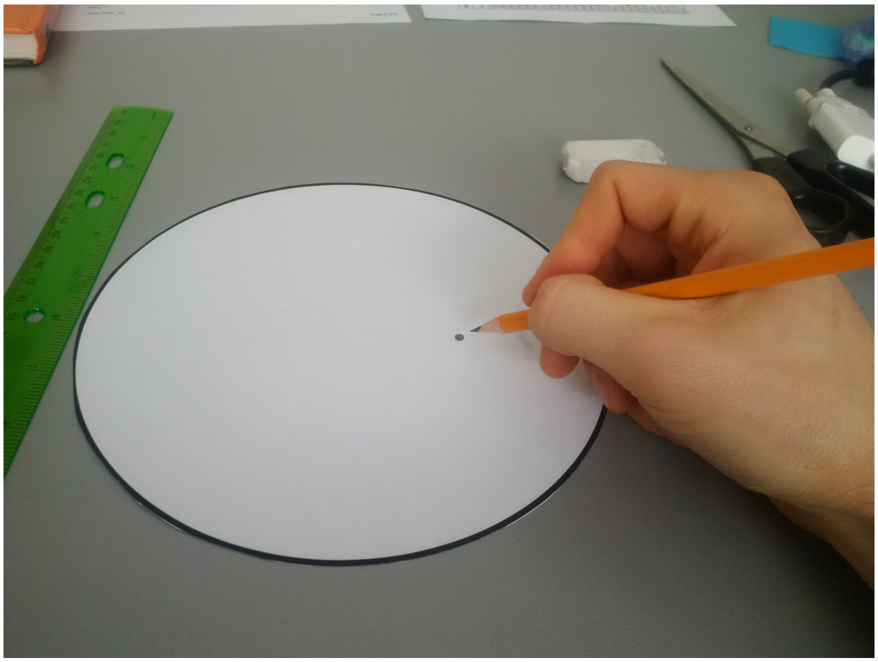
1. What happens when you move the two pins closer and closer to the origin?
2. What shape do you get when you put the knots/pins on top of each other?
3. If you keep the points in the same location, but make the total length of the string shorter, what happens to the shape?

Let's Fold a Circle

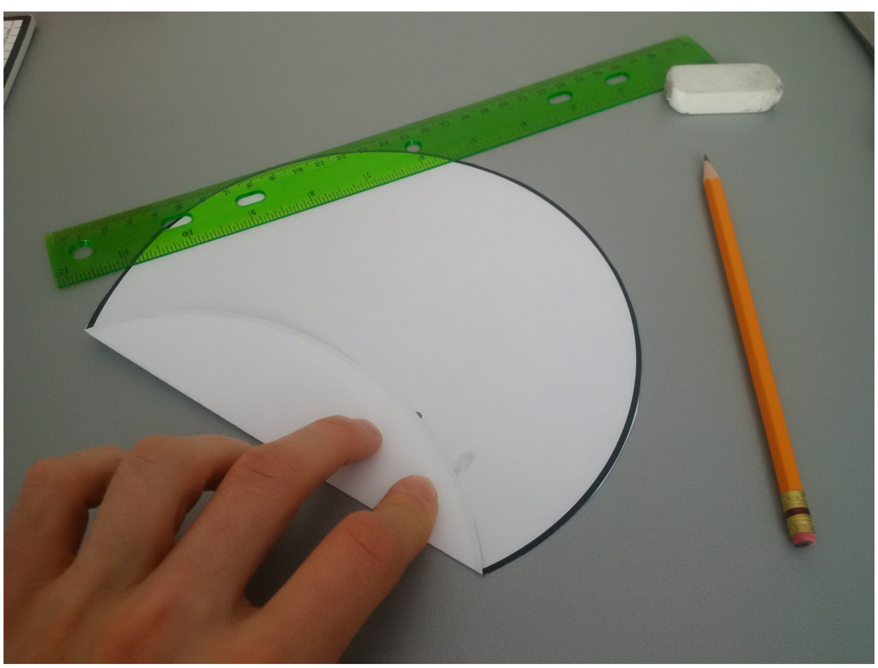
What you will need:

- Paper circle cut out
- Ruler
- Pencil

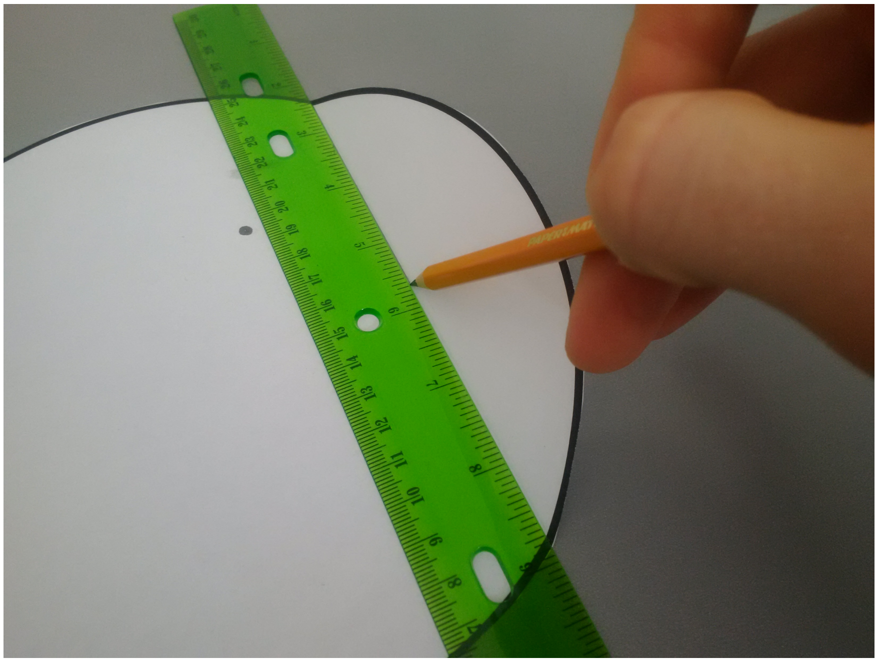
Step 1: Draw a point anywhere inside the circle. Make sure it is large enough to see.



Step 2: Pick a point on the edge of the circle and fold that point over to touch the point you just drew.



Step 3: Draw a line with the ruler along the fold.

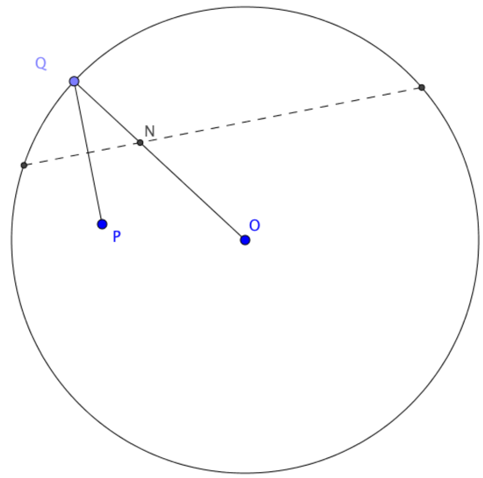


Step 4: Repeat steps 2 and 3 for different points along the edge of the circle. It will likely take 10-12 folds before you start to see a shape

Questions:

1. Can you find the centre of the circle? How can you be sure that's the centre?
2. What shape do you start to see forming around your point and the circle's centre?
3. If you were to do this again with a new paper circle, how might changing the location of the point impact the shape?

Showing this is an Ellipse



Equation of an Ellipse

We will focus on ellipses in standard position.

The **equation of the ellipse** centred at the origin, with foci $(c, 0)$ and $(-c, 0)$ and major axis of length $2a$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a^2 = b^2 + c^2$ and the minor axis has length $2b$.

OR

The **equation of the ellipse** centred at the origin, with foci $(0, c)$ and $(0, -c)$ and major axis of length $2a$ is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

where $a^2 = b^2 + c^2$ and the minor axis has length $2b$.

Practice [On an Ellipse]

For the ellipse $\frac{x^2}{16} + \frac{y^2}{64} = 1$, state

(a) the intercepts

(b) the length of the axes

(c) the foci

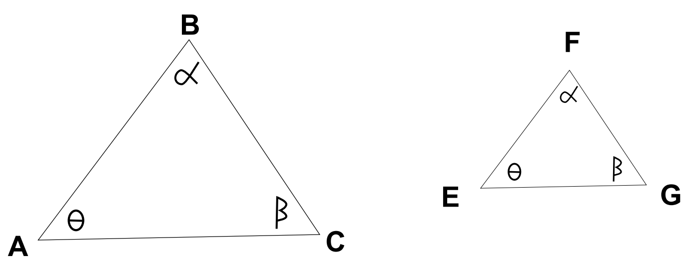
and sketch the ellipse.

Ellipse or Circle?

This "crank" is called **Archimedes Trammel**.

What shape does the handle produce?

Recall [Similar Triangles]:



$$\triangle ABC \backsim \triangle EFG$$

$$\frac{AB}{EF} = \frac{BC}{FG} = \frac{AC}{EG}$$

Conics Sections

Conics Sketched on March 29:

- A **circle** is a set of points a fixed distance (radius) from a fixed point (centre).
- An **ellipse** is a set of points such that the sum of the distance from each of the two fixed points (foci) is constant.

Today's Conics:

- A **hyperbola** is the set of points such that the absolute value of the difference of the distances from each point to two fixed points, the foci, is constant.
- A **parabola** is a set of points equidistant distant from a fixed point (focus) and a fixed line (directrix).

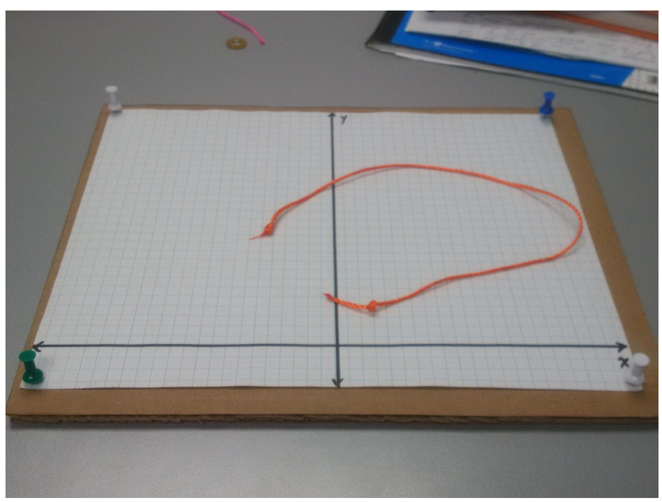
Let's Draw

What you will need:

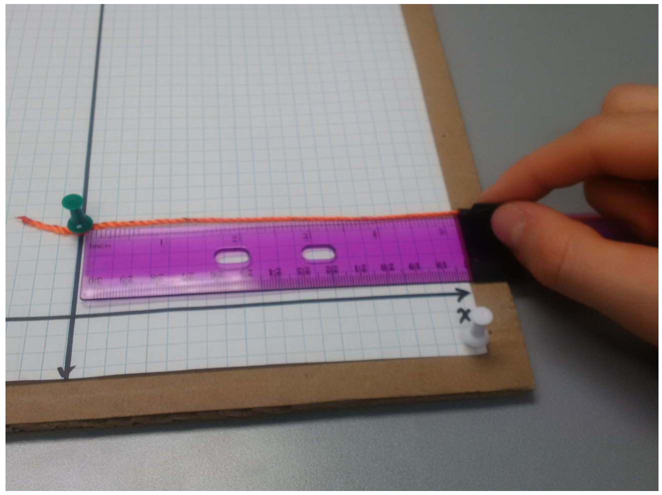
- String
- Cardboard Rectangle
- Tacks x 6
- Ruler
- Paper x 2
- Washer
- Strip of electrical tape

Instructions:

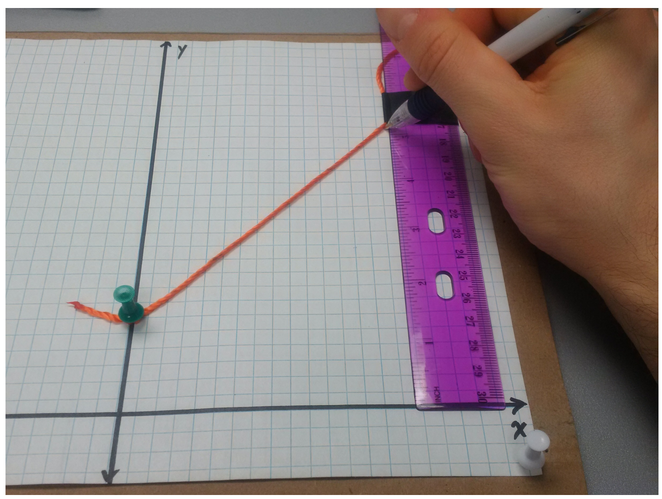
1. Position the pages and boards so the long side is facing you.
2. On one page draw and x- and y-axes on the page to break it into four equal quadrants.
3. On another page draw the x-axis closer to the long edge facing you and the y-axis in the middle of the page.
4. Pin the page with the x-axis closest to the edge to the cardboard so it doesn't move.
5. Tie knots near the ends.



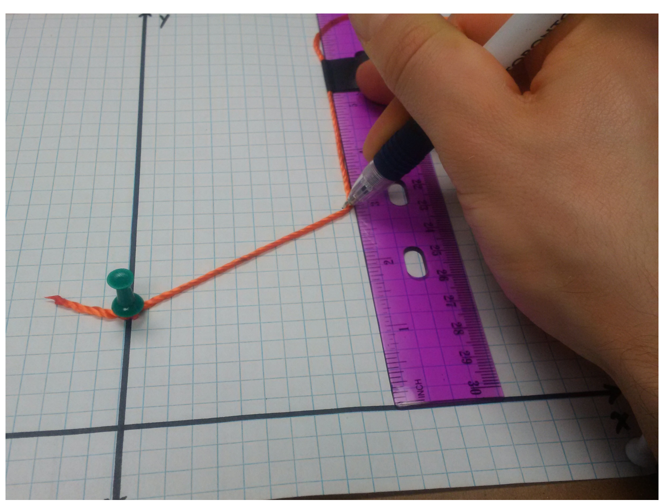
6. Pin one end of a string on the y-axis some number of squares above x-axis
7. Pull the string so it is parallel to the x-axis and set the ruler beside the string, but between the string and the x-axis.
8. Take the strip of electrical tape and tape the ruler to the string where it meets the end of the page.



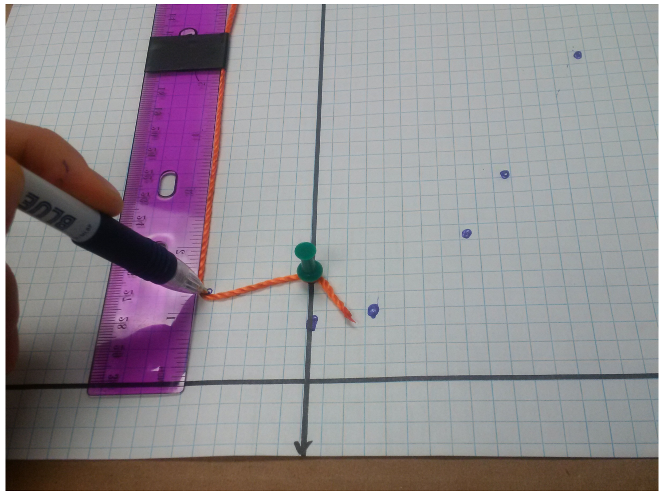
9. Place the ruler so the string is taunt and the ruler is perpendicular to the x-axis.
10. Mark the point where the ruler and string meet.



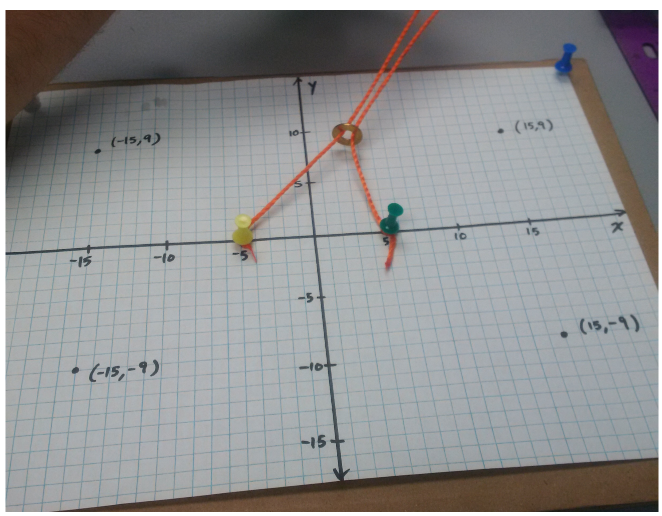
11. Move the ruler closer to the y-axis while keeping it perpendicular to the x-axis.
12. Push the string against the ruler so it is taunt. Mark the initial point where the ruler and string meet.



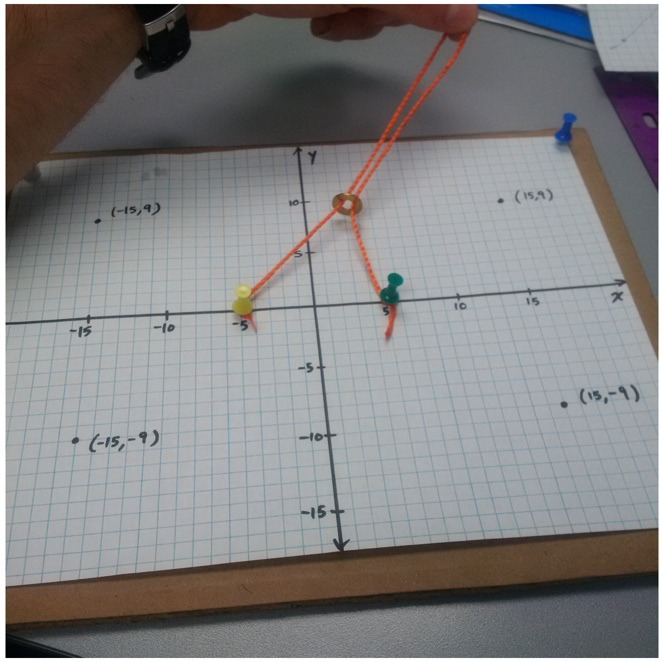
13. Move the ruler closer and closer to the y-axis. At each move plot another point (I.e. Repeat steps 8 &9)
14. Once you reach the y-axis flip the ruler over to Quadrant 2 and repeat step 10, except you are now moving away from the y-axis.



15. Take the string off the ruler and change the paper to the four equal quadrant paper
16. Pin the knots at both ends to the x-axis so they are equal distance from the origin

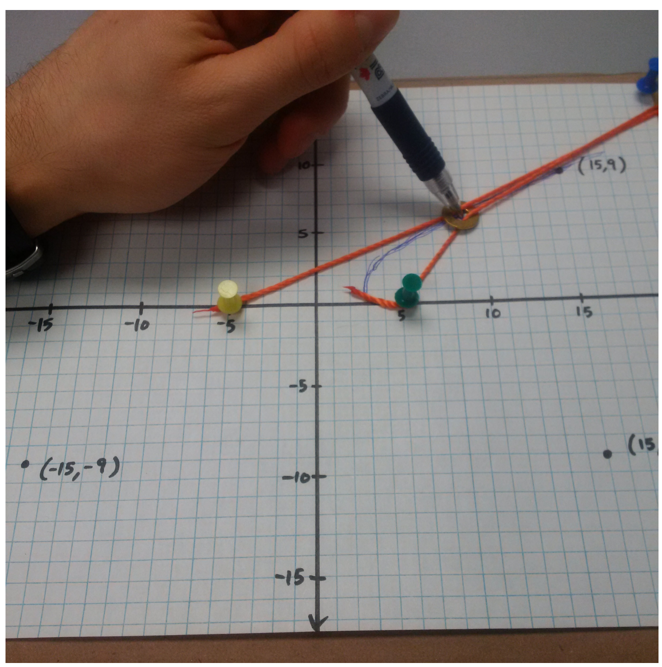


17. Place a point somewhere near the top right corner of Quadrant 1. Mirror this point over the x- and y -axes. Also mirror the mirrored point in Quadrant 4 over the y-axis.
18. Pinch the string and thread the washer through the pinched string (see below)



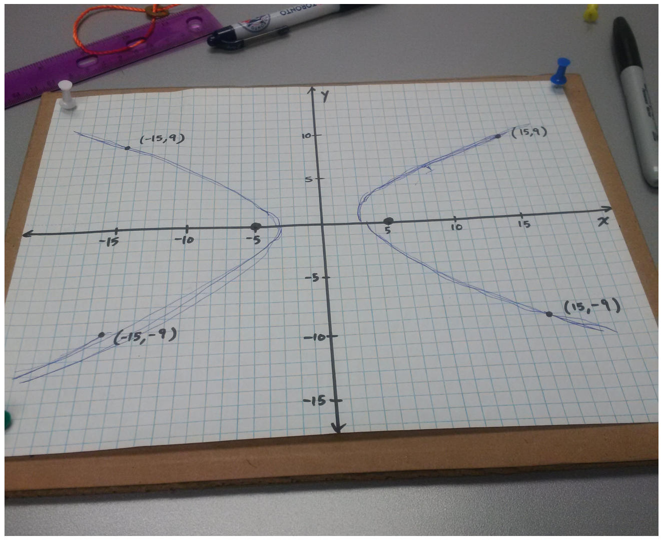
19. Place the washer over one of your points and put your pen or pencil through the washer on to the paper.

20. Continuing pinching the string and move the pen/pencil and washer down the string until you reach the x-axis.



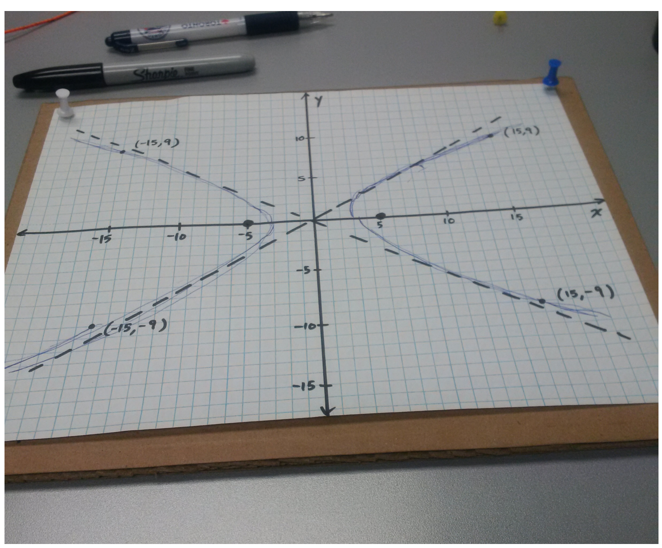
21. Repeat steps 5 and 6 for the three other points.

22. Your final curves should look "roughly" like what is shown below.



Questons

1. What do you think will happen to the curve if you move the point you mirror closer to the x-axis?
2. How is the hyperbola similar to the ellipses you sketched last week?
3. Where do you think the vertices of the hyperbola are located?
4. Can you draw symmetric lines through the origin that are close and get closer as the curve as the curve goes away from the origin, but never never touch the curve?
5. Do you know what these lines are called?



Parabola Facts

- Parabolas open up, down, left, or right
- A **parabola** is a set of points equidistant distant from a fixed point (focus) and a fixed line (directrix).
- The line through the focus and perpendicular to the directrix is called the **axis of symmetry**.
- The point on the axis of symmetry halfway between the directrix and the focus is called the **vertex**.
- The distance from the vertex to the focus is called the **focal length**
- For now the parabolas we will focus on will be in standard position.

Equation of a Parabola

Find the equation of the parabola with its focus at $F(0, c)$ where $c \in \mathbb{R}$ and a directrix at $y = -c$. Again, we will focus on parabolas in standard position.

The general **equation of a parabola** with the y-axis as the axis of symmetry and a focus $(0, c)$ is $x^2 = 4cy$.

OR

The general **equation of a parabola** with the x-axis as the axis of symmetry and a focus $(c, 0)$ is $y^2 = 4cx$.

Where is the vertex?

Practice [World's Most Famous Parabola]

For the parabola $x^2 = y$, state

(a) the vertex and the focus

(b) the direction of the opening

(c) the equation of the directrix

and sketch the parabola.

Equation of a Parabola

For the parabola $x^2 = 4cy$, what if we moved the vertex to $(3, 1)$?

The **equation of a parabola** with vertex (h,k) has $x = h$ as its axis of symmetry and a focus $(h, c + k)$ is $(x - h)^2 = 4c(y - k)$.

OR

The **equation of a parabola** with vertex (h,k) has $y = k$ as its axis of symmetry and a focus $(c + h, k)$ is $(y - k)^2 = 4c(x - h)$.

Practice [Not So Famous Parabola]

For the parabola $(y+1)^2 = -2x$, state

(a) the vertex and the focus

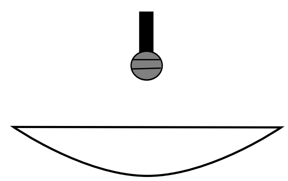
(b) the direction of the opening

(c) the equation of the directrix

and sketch the parabola.

Example [Don't Dish It Out If You Can't Take It!]

A parabolic dish for picking up sound has diameter 1.5 m and is 20 cm deep.



How high above the centre of the dishes base should the microphone be placed to pick up the sound that is reflected to the focus? Round to the nearest centimetre.

History of Conic Sections

- Apolonius of Perga (c. 225 B.C.) discovered that by intersecting a right circular cone with a plane, he could form different shapes

Position of Plane	**Resulting Curve**
perpendicular to the axis of the cone	circle
cutting one part only and not parallel to the generating line	ellipse
parallel to the slant	parabola
cutting the cone parallel to the axis of the cone*	hyperbola*

*There is more to the hyperbola than described, we just don't have time

- Pappus of Alexandria (c. A.D. 300) redefined each of these curves in two dimensions as the locus of a point such that the ratio of its distance from a fixed point (focus) to a fixed line (directrix), is constant.

- Ren Descartes and other analytic geometers (c. 1637) developed two-variable equations from these definitions.