Intermediate Math Circles
Wednesday, April 5, 2017
Problem Set 8

1. Determine the coordinates of the vertices and foci for each of the following ellipses.

   (a) $x^2 + 9y^2 = 36$

   We want equation to be of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
   Manipulate the given equation to match the general equation of an ellipse.

$$\frac{x^2}{36} + \frac{9y^2}{36} = \frac{36}{36}$$
$$\frac{x^2}{6^2} + \frac{y^2}{2^2} = 1$$

   Since $6 > 2$, we know major axis is a horizontal line on the x-axis.
   Therefore the vertices are $(6, 0)$ and $(-6, 0)$.
   We know $a = 6$ and $b = 2$ and that the foci are on the x-axis (i.e. $F(\pm c, 0)$).
   Recall for an ellipse $a^2 = b^2 + c^2$.

$$\begin{aligned} 6^2 &= 2^2 + c^2 \\ c^2 &= 36 - 4 \\ c^2 &= 32 \\ c &= 4\sqrt{2} \end{aligned}$$

   Therefore the foci are $(4\sqrt{2}, 0)$ and $(-4\sqrt{2}, 0)$.

   (b) $25x^2 + 16y^2 = 400$

   Manipulate the given equation to match the general equation of an ellipse.

$$\frac{25x^2}{400} + \frac{16y^2}{400} = \frac{400}{400}$$
$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$
$$\frac{x^2}{4^2} + \frac{y^2}{5^2} = 1$$

   Since $5 > 4$, we know major axis is along the y-axis and foci are $F(0, \pm c)$.
   Therefore vertices are $(0, 5)$ and $(0, -5)$.
   Find c to find foci.

$$\begin{aligned} a^2 &= b^2 + c^2 \\ 5^2 &= 4^2 + c^2 \\ c^2 &= 25 - 16 \\ c^2 &= 9 \\ c &= 3 \end{aligned}$$

   Therefore, the foci are $(0, 3)$ and $(0, -3)$.

(c) $2x^2 + 3y^2 = 12$

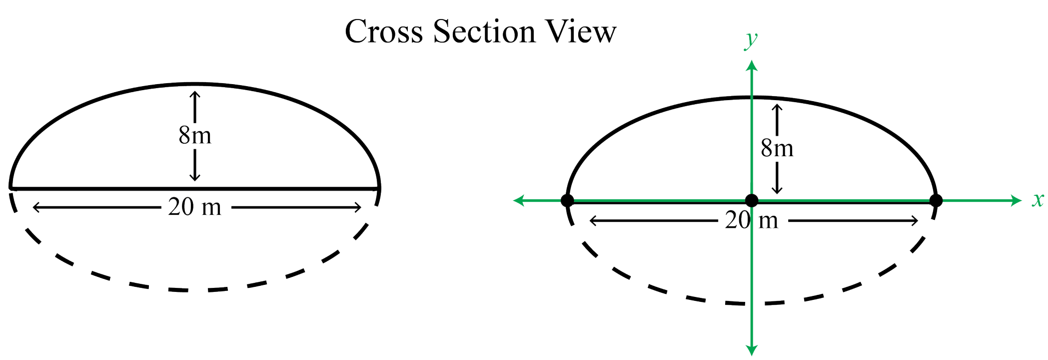
Manipulate the given equation to match the general equation of an ellipse.

$$\begin{aligned} \frac{2x^2}{12} + \frac{3y^2}{12} &= \frac{12}{12} \\ \frac{x^2}{6} + \frac{y^2}{4} &= 1 \\ \frac{x^2}{(\sqrt{6})^2} + \frac{y^2}{2^2} &= 1 \end{aligned}$$

Since $\sqrt{6} > 2 = \sqrt{4}$, we know major axis is along the x-axis and foci are $F(\pm c, 0)$.
Therefore, vertices are $(\sqrt{6}, 0)$ and $(-\sqrt{6}, 0)$.
Find c to find foci.

$$\begin{aligned} a^2 &= b^2 + c^2 \\ (\sqrt{6})^2 &= 2^2 + c^2 \\ c^2 &= 6 - 4 \\ c^2 &= 2 \\ c &= \sqrt{2} \end{aligned}$$

2. A bubble over a set of tennis courts is semi-elliptical in cross-section, 20m wide at the ground, and 8 m high in the centre. How high is the bubble at a point 2m in from the outer edge?



Drop co-ordinates onto the image.

Major axis is along the ground since 20 m $> 2 \times 8$ m $= 16$ m.

Vertices located at $(10, 0)$ and $(-10, 0)$.

Equation of our elliptical bubble is $\frac{x^2}{10^2} + \frac{y^2}{8^2} = 1$.

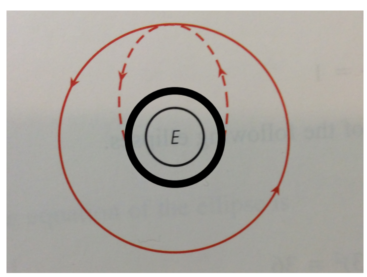
2 m from outer edge has co-ordinates $(8, 0)$ and $(-8, 0)$.

Find the corresponding y-value when $x = \pm 8$.

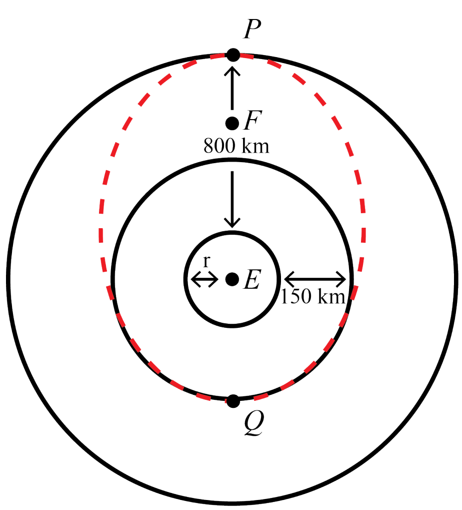
$$\begin{aligned}
\frac{(\pm 8)^2}{10^2} + \frac{y^2}{8^2} &= 1 \\
\frac{64}{100} + \frac{y^2}{64} &= 1 \\
\frac{y^2}{64} &= 1 - \frac{64}{100} \\
\frac{y^2}{64} &= \frac{36}{100} \\
y^2 &= \frac{36(16)}{25} \\
y^2 &= \frac{576}{25} \\
y^2 &= \pm\sqrt{\frac{576}{25}} \\
y &= \pm\frac{24}{5} = \pm 4.8
\end{aligned}$$

Therefore, the bubble is 4.8m high at a point 2m from the bubble's outer edge.

3. A spacecraft is in a circular orbit 800km above Earth. To transfer the craft to a lower circular orbit 150km above Earth, the spacecraft must be placed in an elliptical orbit as shown with centre of Earth at one focus.



Find an equation of the transfer orbit if the radius of the earth is 6336km.

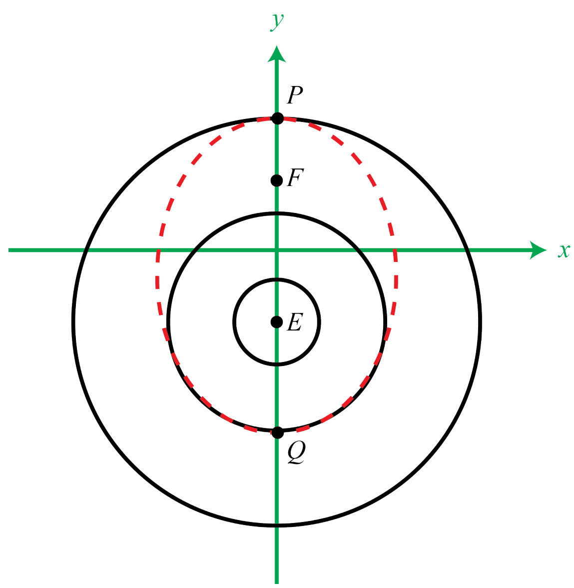


Let P be the point of departure from the original orbit (i.e. one of the vertices).

Let F be the other foci.

Let Q be the vertex opposite P on the elliptical orbit.

Drop the co-ordinates so the origin is at the centre of the ellipse.



We are given the radius of the earth, $r = 6336$ km.

$PE + EQ = 2a \qquad \longleftarrow$ length of major axis.

$$\begin{aligned}(r+800)+(r+150) &= 2a\\ a &= \frac{2r+950}{2}\\ a &= \frac{2(6336)+950}{2}\\ a &= 6811\end{aligned}$$

Thus, according to our co-ordinates $P(0, 6811)$ and $Q(0, -6811)$.

We know E is $(r + 150)$ from Q.

$$\text{Thus,} \qquad \begin{aligned} c &= 6811 - (6336 + 150)\\ c &= 325\end{aligned}$$

Thus, $E(0, -325)$ and $F(0, 325)$.

Find b.

$$\text{Recall:} \qquad \begin{aligned} a^2 &= b^2 + c^2\\ b^2 &= 6811^2 - 325^2\\ b &= \sqrt{6811^2 - 325^2}\\ b &\approx 6803\end{aligned}$$

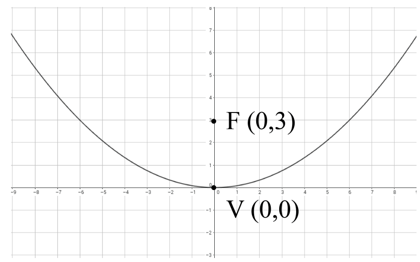
Therefore, the equation of the transfer orbit is *roughly* $\dfrac{x^2}{6803^2} + \dfrac{y^2}{6811^2} = 1$

4. Determine the equation for each of the following parabolas.

(a) vertex $(0, 0)$, focus $(0, 3)$
$V(0, 0)$ and $F(0, 3)$ tells us the parabola opens up and is in standard position.

$$\begin{aligned} x^2 &= 4cy \\ c &= 3 \\ x^2 &= 4(3)y \\ x^2 &= 12y \end{aligned}$$

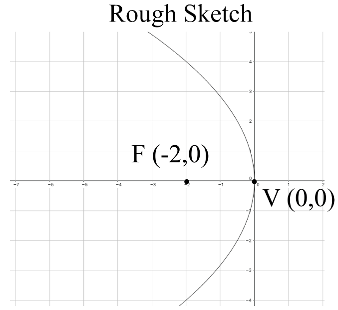
Rough Sketch



(b) vertex $(0, 0)$, focus $(-2, 0)$
$F(-2, 0)$ is to the left of $V(0, 0)$. Therefore, the parabola opens to the left.
$V(0, 0)$ tells us the parabola is in standard position.

$$\begin{aligned} y^2 &= 4cx \\ c &= -2 \\ y^2 &= 4(-2)x \\ y^2 &= -8x \end{aligned}$$

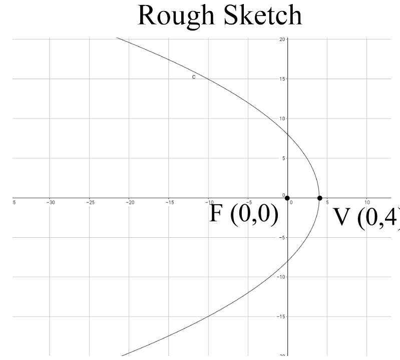
Rough Sketch



(c) vertex $(4, 0)$, focus $(0, 0)$
$V(4, 0)$ is to the right of $x = 1$, meaning the parabola opens right and is **not** in standard position.
This tells us the equation is of the form:

$$\begin{aligned} (y-k)^2 &= 4c(x-h) \\ \text{where} \quad h = 4 &\text{ and } k = 0 \\ y^2 &= 4c(x-4) \\ \text{This tells us } &F(c+h, k) \\ \\ (c+4, 0) &= (0, 0) \\ \implies \quad c &= -4 \\ y^2 &= 4(-4)(x-4) \\ y^2 &= -16(x-4) \end{aligned}$$

Rough Sketch

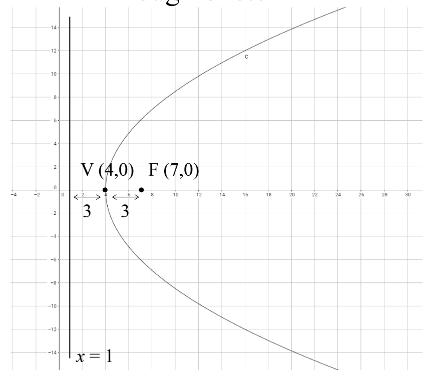


(d) vertex $(4, 0)$, directrix $x = 1$
$V(4, 0)$ is to the right of $x = 1$, meaning the parabola opens right and is **not** in standard position.
This tells us the equation is of the form:

$$(y-k)^2 = 4c(x-h)$$
where $h = 4$and $k = 0$
$$y^2 = 4c(x-4)$$

We know $F(c+h,k) = (C+4,0)$.
The distance from $x = 1$ to V must equal VF and F is on the axis of symmetry $y = 0$.
Thus, $F(7,0)$ and $c = 3$.
Therefore, $y^2 = 12(x-4)$.

Rough Sketch



(e) focus $(0,2)$, directrix $y = -4$

$F(0,2)$ is above directrix and therefore opens up.

We know the vertex is halfway between F and $y = -y$. So find the mid-point of F and the point of intersection between $y = -4$ and $x = 0$ (i.e. $(0,4)$).

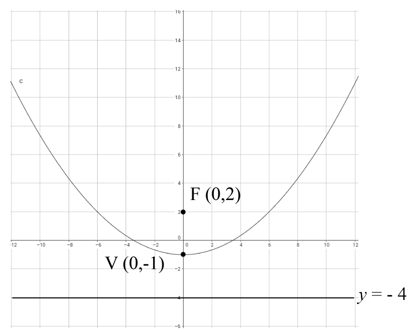
$$M\left(\frac{0+0}{2}, \frac{2+(-4)}{2}\right) = (0,-1)$$

For this parabola we know $F(h,k+c)$.

We also know $h = 0$ and $k = -1$

$$F(0,c-1) = (0,2)$$
$$c-1 = 2$$
$$c = 3$$

Rough Sketch



Therefore, the equation for our parabola is $x^2 = 4(3)(y+1) = 12(y+1)$.

5. Determine the coordinates of the vertex and focus for each of the following:

(a) $y^2 = x$

$V(0,0)$ by inspection of the equation.

Find c

We want c so that $y^2 = x$ is equivalent to $y^2 = 4cx$.

Meaning we want:

$$1 = 4c$$
$$c = \frac{1}{4}$$
$$F\left(\frac{1}{4}, 0\right)$$

(b) $x^2 = -y$

$V(0,0)$ by inspection of the equation.

Find c

We want $x^2 = -y \iff x^2 = 4cy$. (Note: $\iff$ means equivalent).

Meaning we want:

$$-1 = 4c$$
$$c = -\frac{1}{4}$$
$$F\left(-\frac{1}{4}, 0\right)$$

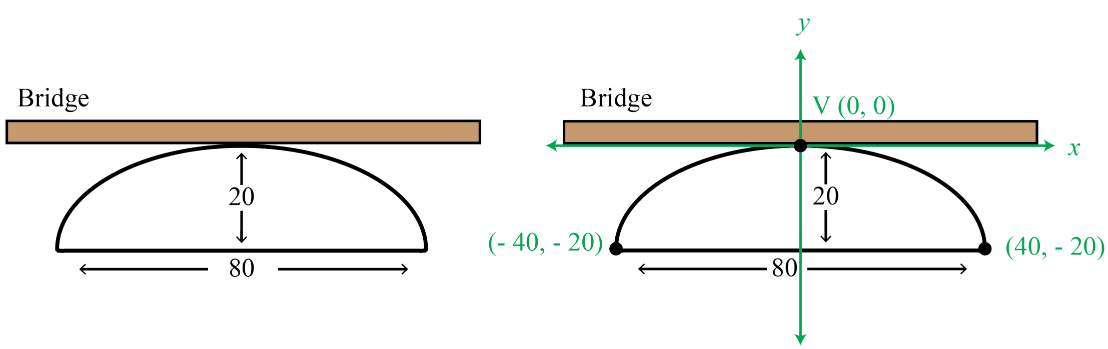
(c) $(x-3)^2 = 4y$
$V(3,0)$ by inspection of the equation.
Thus, $h = 3$ and $k = 0$.
Find c
We want $(x-3)^2 = 4y \iff (x-h)^2 = 4c(y-k)$.
Meaning we want:

$$4 = 4c$$
$$c = 1$$
$$F(h, c+k) = (3, 1+0) = (3, 1)$$

(d) $(y+1)^2 = -12x$
$V(0,-1)$ by inspection of the equation.
Thus, $h = 0$ and $k = -1$.
Find c
We want $(y+1)^2 = -12x \iff (y-k)^2 = 4c(x-h)$
Meaning we want:

$$-\frac{12}{4} = \frac{4c}{4}$$
$$c = -3$$
$$F(h+c, k) = (0-3, -1) = (-3, -1)$$

6. A parabolic arch is used to support a bridge. The arch is 80m wide at the base and the height of the vertex is 20m. How high is the arch at a point 20 m in from either end of the base?



Place origin at vertex of the bridge's parabolic arc. Find the equation of the parabola. We know it is of the form:

$$x^2 = 4cy$$
$$\text{Sub in } (40, -20)$$
$$40^2 = 4c(-20)$$
$$1600 = -80c$$
$$c = -20$$
$$x^2 = -80y$$

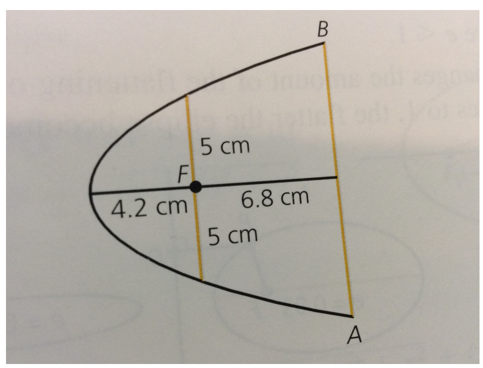
By construction, we know $(-20, y)$ and $(20, y)$ are the points from either end of the 20m base.

Sub one of the points into $x^2 = -80y$ and solve for y.

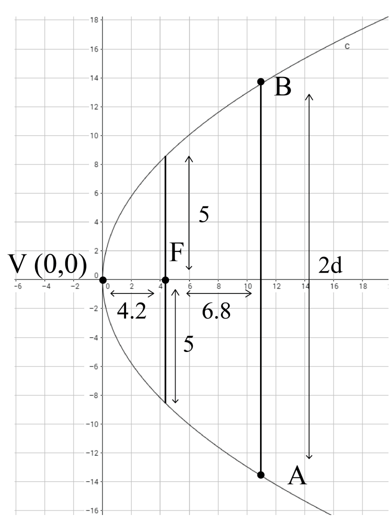
$$(\pm 20)^2 = -80y$$
$$400 = -80y$$
$$y = -5$$

Therefore, the arch is 20m - 5m = 15m high 20m from either base.

7. The cross-section of a parabolic reflector is as shown. The bulb is located at the focus, F.



Find the diameter AB of the opening.



Place the origin at the vertex.

Then, by construction, our parabolic reflection has $F(4.2, 0)$, telling us $c = 4.2$.

$$\text{Thus,} \quad \begin{aligned} y^2 &= 4(4.2)x \\ y^2 &= 16.8x \end{aligned}$$

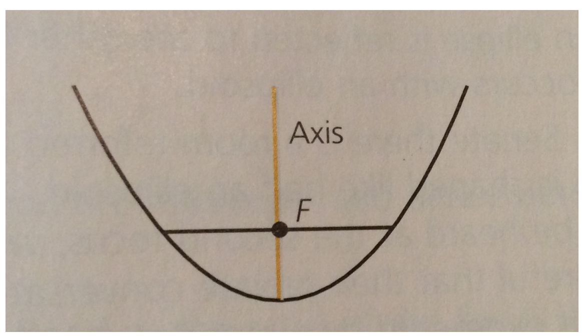
Let the length of $AB = 2d$. Now by construction, $B(11, d)$ and $A(11, -d)$.

Sub B into equation:

$$\begin{aligned} 11^2 &= 16.8d \\ d &= \frac{121}{16.8} \\ 2d &= 2(\frac{121}{16.8}) \\ 2d &= \frac{121}{8.4} \approx 14.4 \end{aligned}$$

Therefore, the diameter of opening AB is 14.4 cm.

8. The **latus rectum** for a conic section is defined as the chord through the focus perpendicular to the axis.



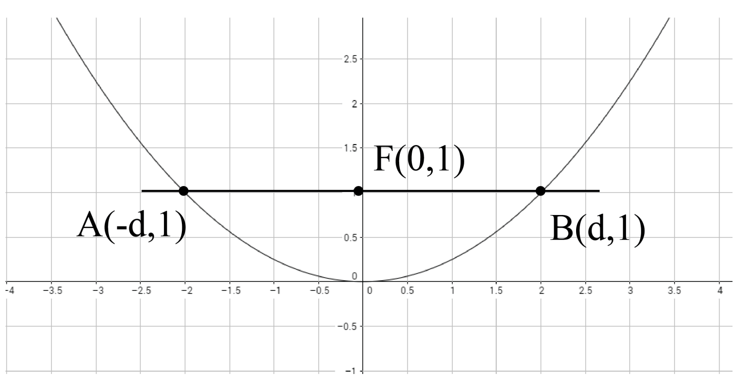
Find the length of the latus rectum for each of the following

(a) $x^2 = 4y$

$x^2 = 4y \iff x^2 = 4cy.$

This tells us $c = 1$ and $F(0, 1)$.

Rough Sketch



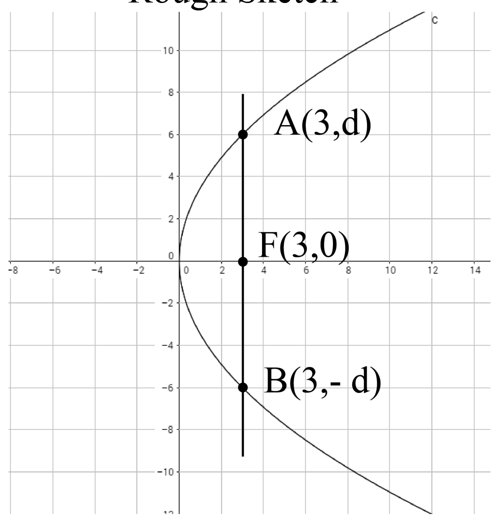
Sub $y = 1$ into equation and solve for x.

$$\begin{aligned} x^2 &= 4(1) \\ x &= \pm 2 \\ \text{Thus,} \quad d &= 2 \end{aligned}$$

Therefore, the length of latus rectum is 4.

(b) $y^2 = 12x$
$y^2 = 12x \iff x^2 = 4cy$
Tells us $12 = 4c$ or $c = 3$ and $F(3, 0)$.

Rough Sketch



Sub $x = 3$ into equation and solve for y.

$$\begin{aligned} y^2 &= 12(3) \\ y^2 &= 36 \\ y &= \pm 6 \end{aligned}$$

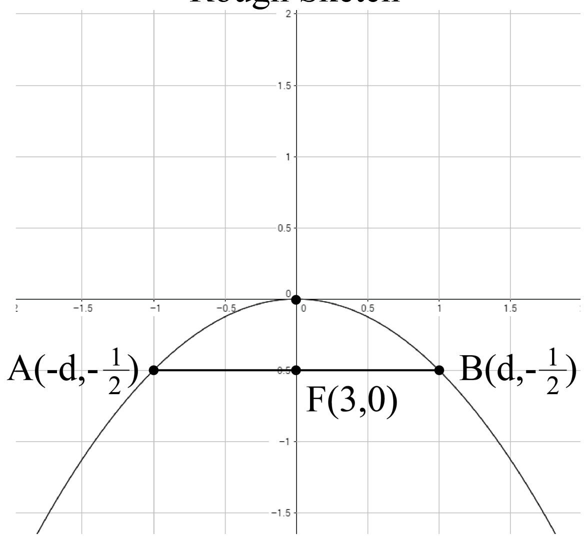
Thus, $d = 6$. Therefore the length of latus rectum is 12.

(c) $x^2 = -2y$

$x^2 = -2y \iff x^2 = 4cy.$

This tells us $-2c = 4c$ or $c = -\frac{1}{2}$ and $F\left(0, -\frac{1}{2}\right)$.

Rough Sketch



Sub $y = -\frac{1}{2}$ into equation and solve for x.

$$x^2 = -2\left(-\frac{1}{2}\right)$$
$$x^2 = 1$$
$$x = 1$$

Therefore, the length of latus rectum is 2.

9. 2012 Hypatia Question 3
   For the question and solution go to *cemc.uwaterloo.ca/contests/past_contests.html*

10. 2015 CTMC- Individual Problems Question 9
    For the question and solution go to *cemc.uwaterloo.ca/contests/past_contests.html*