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Centre for Education in
Mathematics and Computing

Grades 7 & 8, Math Circles

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Graph Theory

Solutions

Example 1

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1. To represent our handshake situation as a graph, we represent each person with a vertex (so 5 vertices) and each handshake with an edge. The resulting graph is shown in Figure 1 below.

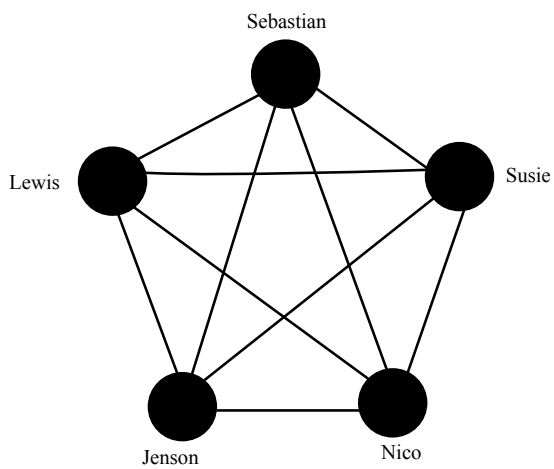

Figure 1: A graph representing the people at the party and the handshakes

2. From our graph, we see that the degree of each vertex is the same i.e. 4. This represents the fact that each person shakes hands with 4 others.

3. We said that each edge represents a single handshake. This means that the number of edges in our graph must be the total number of handshakes that go around.

   From Figure 1, we see that there are 10 edges. This means that there were a total of 10 handshakes at the party.

The key thing to note about this problem is the double counting that happens if we sum up the degrees of each vertex in our graph. The sum of degrees of each vertex is 20. The number of edges is exactly half this number at 10.

Example 2

This question is a bit tricky in that **BOTH** the graphs are planar. You can see this by looking at the graph in Figure 3. You could take the vertices $\{i, h, d, j\}$ and shrink them so that they fit inside the vertices $\{a, c, g, f\}$ (as in Figure 2).

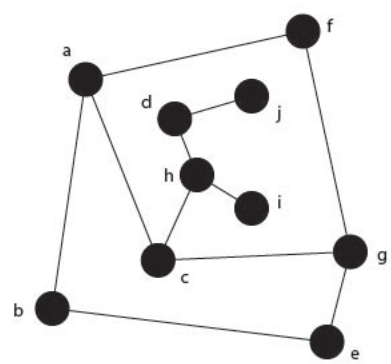

Figure 2: Example 2(a)

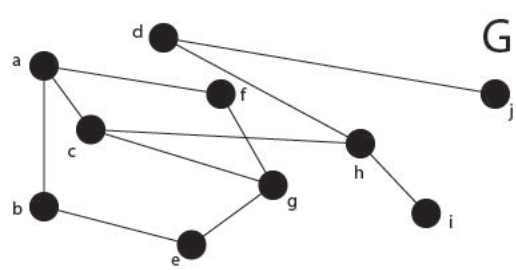

Figure 3: Example 2(b)

Remember, that if a graph can be redrawn in such a way that it is planar, then it has a **planar embedding**.

When talking about graphs, we only care about the vertices and edges. Count the number of vertices and edges for each of the above graphs - you should get the same answer.

Example 3

The question asks you whether the graph shown below (Figure 4) is planar or non-planar, according to Theorem 1.

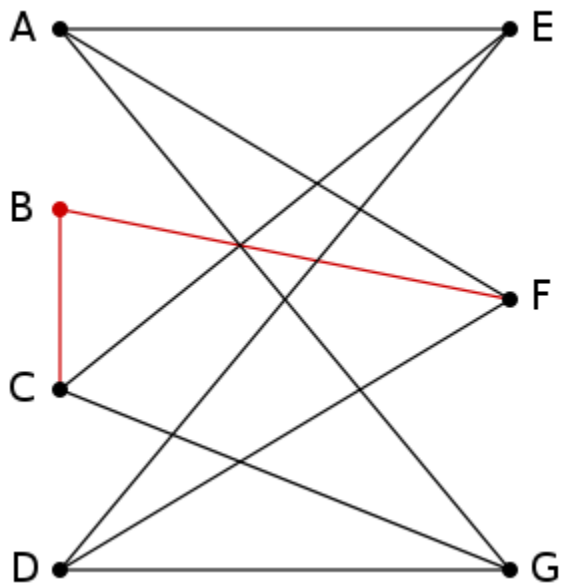

Figure 4: Example 3

Theorem 1 tells us that if a graph with v vertices and e edges satisfies the following condition:

$$e > (3v - 6)$$

it must be non-planar.

The graph shown in Figure 4 has 7 vertices and 10 edges. Our inequality becomes:

$$10 > 21 - 6\ 10 > 15$$

Our inequality is clearly false as 10 is not greater than 15. This means that **Theorem 1 doesn't tell us anything about the planarity of the graph**. Theorem 1 applies only if the condition is true. If the condition is false, then the Theorem doesn't tell us anything.

So the answer to this question would be that Theorem 1 doesn't tell us anything about the planarity of the graph.

It turns out that this graph is actually non-planar.

Example 4

The problem asks you to assign colours to each and every vertex of the graph such that no two adjacent vertices have the same color.

Assigning colors is the same as naming/numbering the vertices. The solution shown in Figure 5 is one such possible 'colouring' for the graph.

Notice that the fewest number of distinct 'colours' we needed was 3.

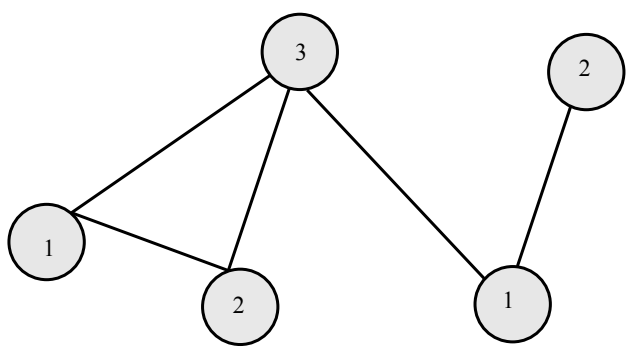

Figure 5: Example 4

Example 5

The question again asks you to find the fewest number of colours needed to colour the entire graph (i.e. the chromatic number).

We can colour the graph as shown in Figure 6 below.

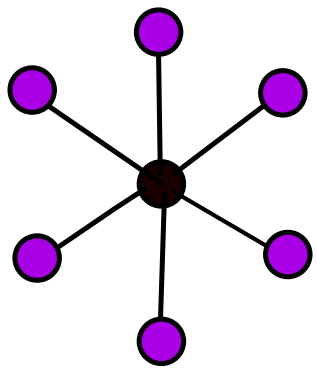

Figure 6: Example 5

Remember that colouring a graph is the same as assigning numbers to each of the vertices such that no two adjacent vertices have the same number. We could've chosen to use numbers (as we did in Example 4) if we wanted to.

This particular graph is interesting because it's an example of a **bipartite** graph. This means that the vertices can be split into exactly two groups such that all edges start from the vertices in one group and end at the vertices in the other.

A bipartite graph always has a chromatic number of 2.

Problem Set

1. (a) Graph G_1 has 9 vertices and 8 edges. So we write $G_1(9, 8)$.

   Graph G_2 has 11 vertices and 13 edges. So we write $G_2(11, 13)$.

   (b) In G_1, the vertex v has the neighbors $\{r, y, s\}$.

   In G_2, the vertex v has the neighbors $\{z, r, s\}$.

   (c) In G_1:

   The vertex u has a degree of 1 i.e. $deg(u) = 1$.

   $deg(v) = 3$

   $deg(r) = 2$

   Similarly, in G_2:

   $deg(u) = 2$

   $deg(v) = 3$

   $deg(r) = 3$

   (d) G_1 has no edges crossing over each other. This means that G_1 is **planar**. However, you can't really verify if G_1 is planar using Theorem 1. To see this, try to see if G_1 satisfies the condition of Theorem 1.

   This gives:

$$8 > [(3 \times 9) - 6]$$
$$8 > 21$$

   Which is clearly false. So Theorem 1 doesn't tell you anything about the planarity of G_1.

   Similarly, G_2 has no edges crossing over each other and so is **planar**. But trying to verify this using Theorem 1 leads to an absurdity:

$$13 > [(3 \times 11 - 6]$$
$$13 > 27$$

   So once again, Theorem 1 doesn't tell you anything about the planarity of G_2.

   The point of this question is for you to see the pitfalls of using Theorem 1 in the wrong place.

2. In G_1, a walk from s to y involves visiting the vertex v exactly once. This means that in G_1, a walk from s to y is a path.

   In G_2, a walk from s to y involves visiting the vertices $\{v, z, w, x\}$ exactly once. So this is also a path.

   However, there are two alternate walks between s and y in G_2 which involve visiting the vertices $\{u, w, x\}$ or the vertices $\{r, b, a, t, x\}$. Both these walks are also paths as each vertex is visited exactly once.

3. The given graph is **bipartite**. This is because the vertices can be split into exactly two groups such that all edges start from the vertices in one group and end at the vertices in the other group (see Figure 7).

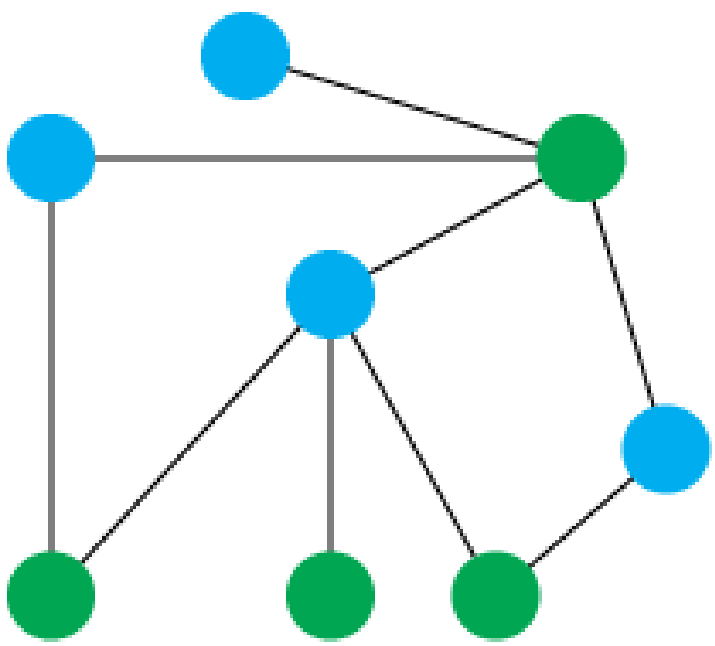

Figure 7: The graph can be coloured using just 2 colours

   The blue vertices are one group (say A) while the green vertices are another (say B). Every edge starts at a blue vertex and ends at a green one (or vice-versa).

4. The colourings are as shown below:

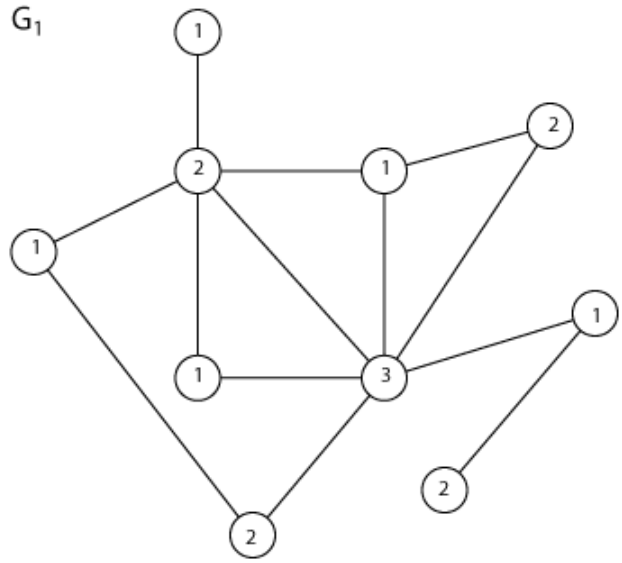

Figure 8: One possible colouring for G_1

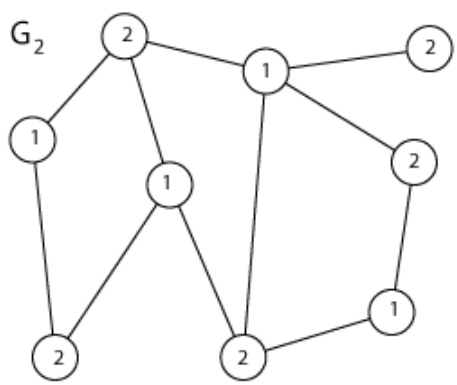

Figure 9: One possible colouring for G_2

G_1 has a chromatic number of 3, whereas G_2 has a chromatic number of 2.

5. One easy way to make a graph out of a map is to put one vertex in each region of the map. We then draw edges between regions that are adjacent. Doing this to the map given in this problem, leads to the following graph (Figure 10).

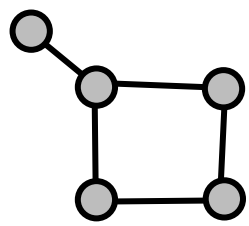

Figure 10: Representing the map as a graph

Notice that the resulting graph is bipartite, so we can colour it using a minimum of 2 colours (Figure 11).

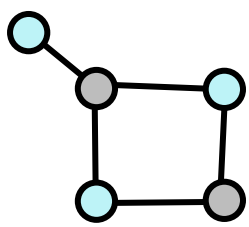

Figure 11: Colouring our graph

This means our map can be coloured by a minimum of 2 colours such that no two adjacent regions have the same colour.

6. The graph in Problem 5 is bipartite. The groups A and B are colour coded as shown in Figure 11 (light blue and grey).

7. The given graph is **non-bipartite**. To see why, try colouring it - you need a minimum of 3 different colours to colour the entire graph i.e. the chromatic number of the graph is 3.

   A bipartite graph can be coloured with just 2 colours.

8. The question asks us to find the chromatic number of the map i.e. the least number of distinct colours needed to colour the entire map such that no two adjacent regions have the same colour.

   To start, we do the same trick as in Problem 5 - we put one vertex in each region of the map and draw edges between adjacent regions on the map. This gives us the following graph (Figure 12).

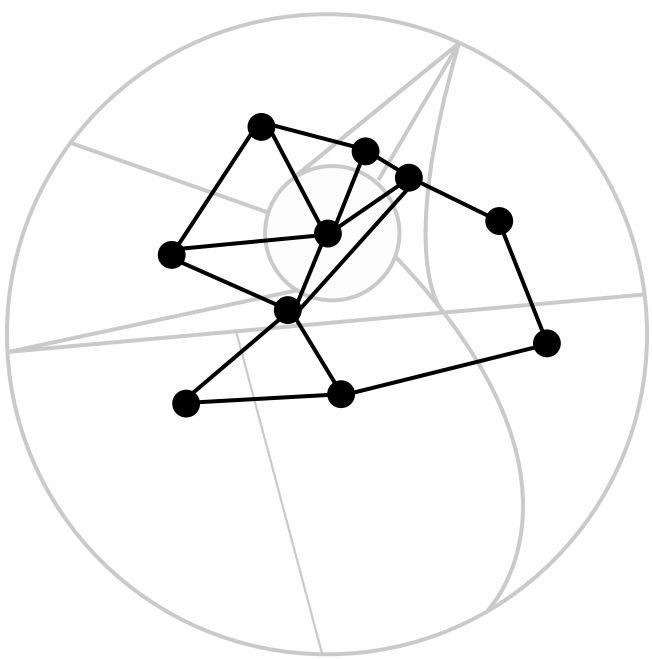

Figure 12: Representing the map as a graph

This graph can be coloured using a minimum of 4 different colours (as shown in Figure 13).

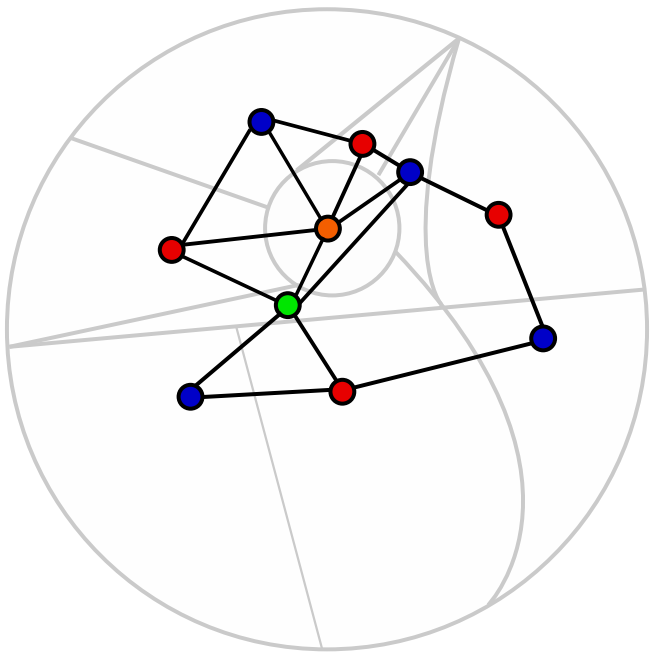

Figure 13: The map can be coloured with a minimum of 4 colours

It's interesting how even complicated maps (such as the one in this problem) can be coloured using at most 4 colours. This fact is called the **Four Colour Theorem**.

9. This is an example of a **Scheduling Problem**. It asks you to find the fewest number of time slots needed to schedule 5 courses without any conflicts.

   We want to represent the problem as a colouring problem. To do this, we represent each course with a vertex. We also represent conflict between courses with an edge between the two vertices representing two conflicting courses. Our edges represent conflicts because conflicting courses cannot be given the same time slot - just like adjacent vertices cannot have the same colour.

The resulting graph is shown in Figure 14.

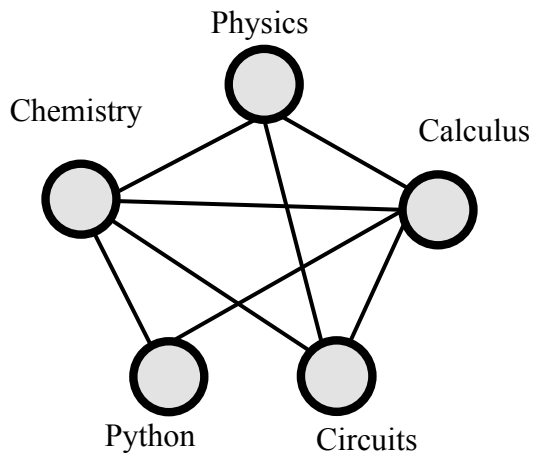

Figure 14: Translating the scheduling problem into a graph colouring problem

The degree of each vertex in the graph corresponds to the number of conflicts the course representing the vertex has. For example, the Python course has conflicts with both Calculus and Chemistry and therefore has edges that connect it to both Calculus and Chemistry.

This graph can be coloured with a minimum of 4 distinct colours. This means that if we were to assign time slots to each course, we'd need a minimum of 4 time slots to ensure that there are no conflicts between courses.

Each time slot is 1 hour long, which means that the University has to be open for at least 4 hours for all courses to be completed without conflicts.