SET THEORY - ASSIGNMENT 2

Functions.

(1) For each function, determine if it is injective, surjective, bijective, or none of the above.
   (a) $f : \{1, 2, 3, 4\} \to \{5, 6, 7\}$, such that $f(1) = 7, f(2) = 6, f(3) = 7, f(4) = 6$.
   (b) $g : \mathbb{Z} \to \mathbb{Z}, g(a) = 2a$.
   (c) $h : \mathbb{N} \to \mathbb{N}$, such that $h(a)$ is the number of prime numbers smaller than a.
   (d) $k : [0, 3] \to [3, 9], k(x) = 2x + 3$.
   (e) $p : \mathbb{R}^3 \to \mathbb{R}^2, p(x, y, z) = (x, y)$.
   (f) $i : A \to \mathcal{P}(A), i(a) = \{a\}$, where A is some set.

(2) Find a function $f : [0, 2] \to [0, 1]$ that is surjective but not injective, a function that is injective but not surjective, a function that is neither, and a function that is both.

(3) Suppose that we have surjective functions $f : A \to B$ and $g : B \to C$. Find a surjective function from A to C.

Cardinalities.

(1) For each of the following pairs of set, determine whether they have the same cardinality. If not, find the bigger set.
   (a) $A =$ the set of functions from $\{1, 2, 3\}$ to $\{1, 2\}$, $B = \{1, 2, 3\} \times \{1, 2\}$.
   (b) $A =$ the students in the class, $B =$ the chairs in the class.
   (c) $A =$ the rational numbers whose denominator is a multiple of 7, $B = \mathbb{Q}$.

(2) In Hilbert's hotel, each room is labeled with a natural number, $0, 1, 2, 3, \ldots$. An infinite group of people comes in, in which every person is labeled with a real number between 0 and 1 with a finite presentation. That is, the sequence of digits after the decimal point doesn't go on forever. Assuming that the hotel was empty before, show that each person can be assigned a different room. What if the hotel was full when the group came in?

(3) Let A and B be sets (either finite or infinite). Show that $|A \cap B| \le |A| \le |A \cup B|$.

(4) Show that for every set A, $|A| \le |\mathcal{P}(A)|$.

(5) If α and β are cardinal numbers (finite or infinite), then $\alpha + \beta$ is defined as follows. Choose sets A and B of cardinality α and β respectively, such that $A \cap B = \emptyset$. Then $\alpha + \beta$ is defined to be the cardinality of $A \cup B$.
   (a) Show that with this definition, $3 + 4 = 7$.
   (b) Show that $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 + \aleph = \aleph$.

Higher cardinalities.

(1) Find an explicit surjection from $\mathcal{P}(\mathbb{N})$ to $\mathbb{N}$.

(2) Find a surjection from $[0, 1]$ to $\mathbb{R}$. Conclude that $|[0, 1]| = \aleph$.

(3) In this question, we will show that $2^{\aleph_0} = \aleph$.

(a) Let $A = \{0,1\} \times \{0,1\} \times \{0,1\} \times \{0,1\}$ (in other words, A consists of sequences of length four of zeros and ones). Find a surjection from A to the set $\{0,1,2,3,4,5,6,7,8,9\}$.
(b) Show that every subset of $\mathbb{N}$ can be represented by an infinite sequence of zeros and ones. Conclude that $\mathcal{P}(\mathbb{N})$ is isomorphic to the collection such sequences.
(c) Use the first two parts to find a surjection from $\mathcal{P}(\mathbb{N})$ to the collection of infinite sequences of $\{0,1,2,3,4,5,6,7,8,9\}$. Conclude that $2^{\aleph_0} \geq |[0,1]|$.
(d) Find a surjection from $[0,1]$ to $\mathcal{P}(\mathbb{N})$, and conclude that $2^{\aleph_0} = |[0,1]|$
(e) Use question **??** to conclude that $2^{\aleph_0} = \aleph$.

(4) Let A be the collection of functions from $\mathbb{R}$ to $\mathbb{N}$. Show that $|A| > \aleph$.

A confusing question.

(1) The purpose of this question is to show that the set of all sets doesn't really exist. Assume that there was a set of all sets, call it S. Show that $|S| > |S|$ (hint: use a similar argument to the one used in the proof that $|\mathcal{P}(A)| > |A|$).