UNIVERSITY OF WATERLOO

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Centre for Education in
Mathematics and Computing

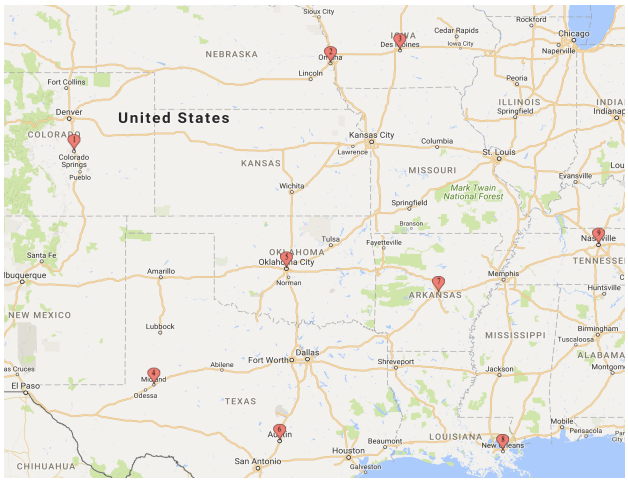
Grade 6 Math Circles

Winter 2018 - February 13/14

Graph Theory - Solutions

Warm-up!

Imagine you are the manager of your favourite sports team or singer. You need to schedule their travel route so that they do as little travel as possible, starting and ending at Colorado Springs. What is the shortest path?



Today we will study **graph theory**. We will learn some definitions in order to figure out what graph theory is!

Definitions

A **vertex** is a point. When we have more than one, we call them **vertices** and denote them by v.

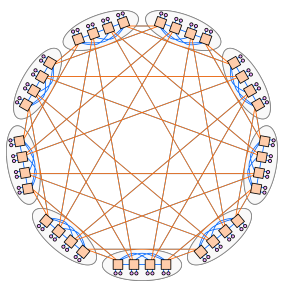
An **edge** is a line segment that joins two vertices, denoted by e.

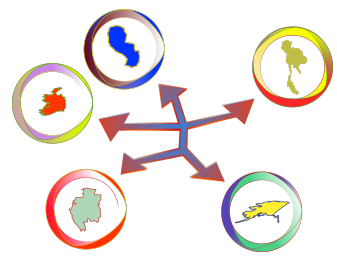
When two vertices are connected by an edge, we say that the vertices are **adjacent**.

The **endpoints** of an edge are the vertices at either end.

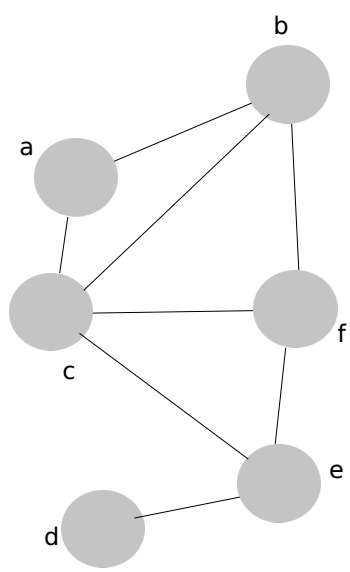
A vertex is **incident** to an edge if it is that edge's endpoint.

A **graph** G is a set of vertices and edges. We write G=(V,E).
Graphs can represent networks (think about a connection of someone's Facebook friends!), a map or road system (see the warm-up), or any connection between different objects!



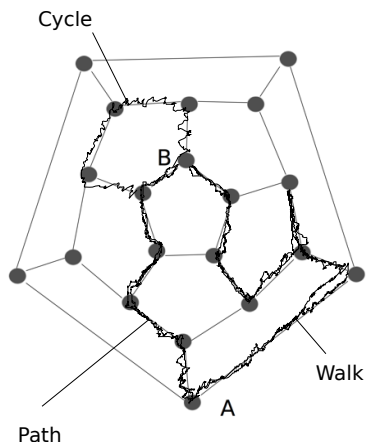


The **neighbourhood** of a vertex is all of the vertices adjacent to that vertex. Two vertices in a neighbourhood are called **neighbours**.
From the following graph, highlight the edges and circle the vertices. List vertices that are adjacent with each other. Find the neighbourhood of the vertex *f*.



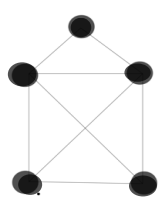
Edges (and adjacent vertices: ab, ac, bf, bc, ce, fe, ed
Vertices: a, b, c, d, e, f
Neighbourhood of f: {b, c, e}
A **walk** is a sequence of vertices where each vertex in the walk is adjacent.
A **path** is a walk without repeated vertices.

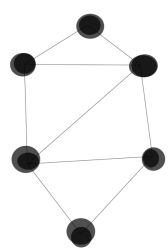
A **cycle** is a path that starts and ends at the same vertex.
From the following graph, find a walk from A to B and a path from A to B.



Planarity

A **planar** graph is a graph that can be drawn such that no edges cross each other.
Here are some examples of planar graphs:

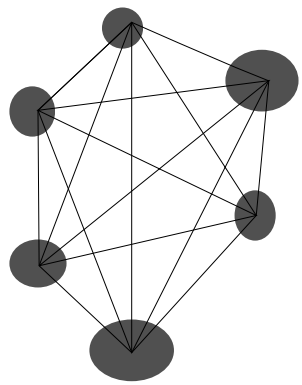




What do you notice about the first planar graph?
There are edges that cross each other!
A planar graph can sometimes be drawn with edge crossings - it is planar as long as there is some way that it can be drawn without crossings!
Now take note of the number of vertices and edges for each planar graph.
5 vertices, 8 edges; 6 vertices, 9 edges.

A graph is not planar if the following result holds:

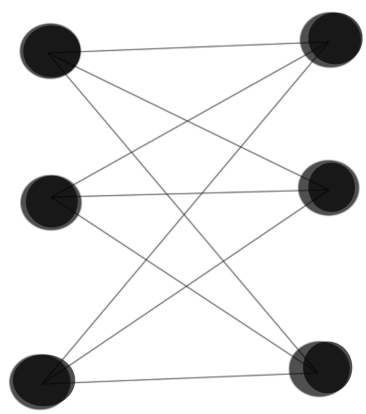
Number of edges > 3v - 6



This doesn't necessarily tell us whether a graph is planar!

(It tells us when a graph is **not** planar)

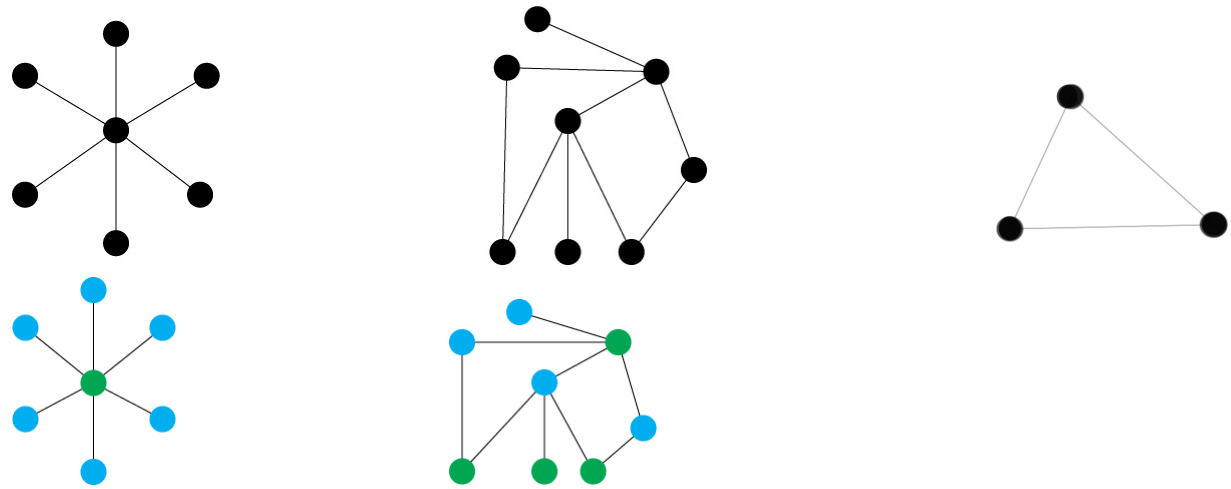
For example, the following graph is not planar, but it does not satisfy the above inequality!



Bipartite Graphs

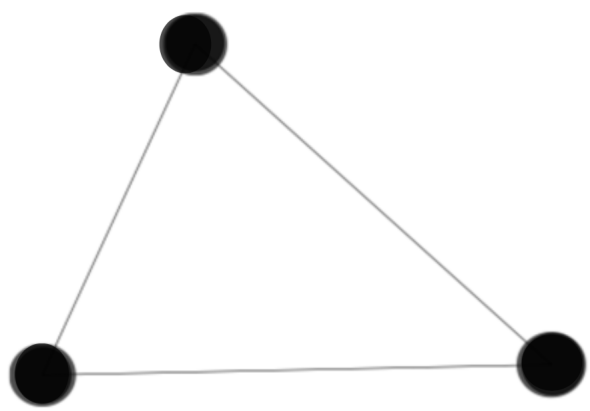
We say that a graph is **bipartite** if the vertices of the graph can be split into two sets, where no two vertices from either set are adjacent.

Are the following graphs bipartite? If so, label the vertices of one set "A" and the vertices of the other "B". If not, explain why.



This graph is not bipartite since we would need 3 different letters to label the vertices.

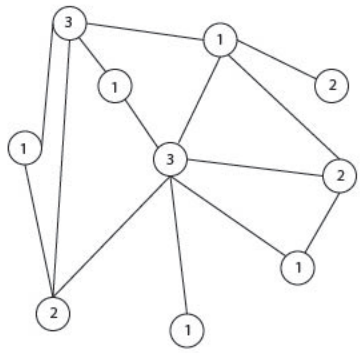
Draw a planar, non-bipartite graph.

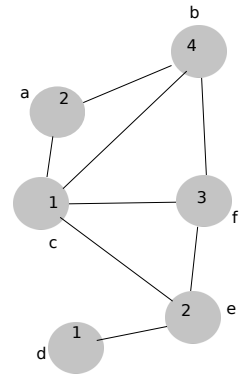


Colouring

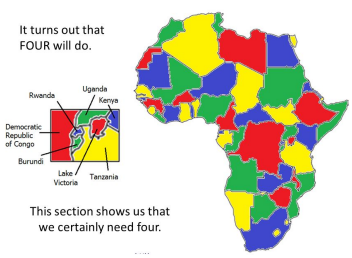
When we **colour** a graph or give it an **n-colouring** (n denoting the minimum number of colours needed for the graph), we label the vertices so that no two adjacent vertices share the same colour.

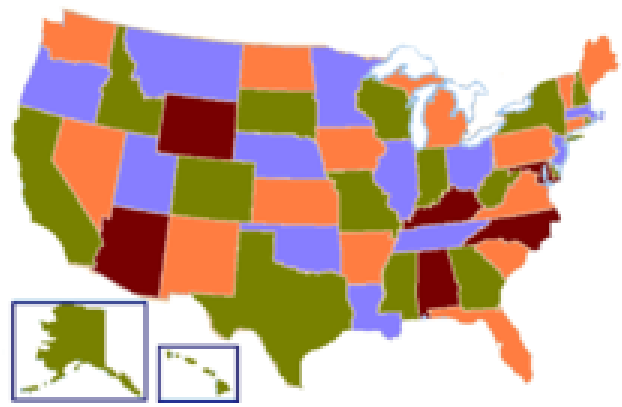
Recall that we can colour bipartite graphs using just 2 colours! So **bipartite graphs are 2-colourable**. Colour the following graphs:





Notice that maps are just graphs without vertices: they are connected at the edges!
Now colour the following maps. Use the least amount of colours possible.





What is the least amount of colours we can use to colour a map?
The 4-colour theorem states that we can colour any map using at most 4 colours!

Weighted Graphs and Minimum Spanning Trees

A **weighted graph** is a graph that assigns a numerical weight to each edge.

If you have ever played the game *Ticket to Ride*, you can think of weights as the number of trains between each city.

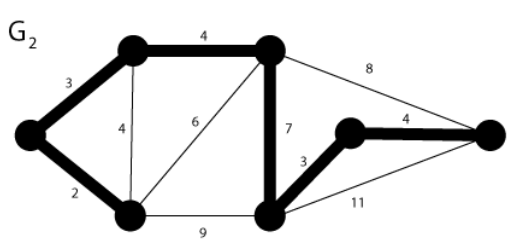
Retrieved from: https://play.google.com/store/apps/details?id=com.daysofwonder.tt.android

A **tree** is a graph with no cycles.

A **spanning tree** is a tree that contains all vertices of a graph.

A **minimum spanning tree** is a spanning tree that has the smallest total weight (found by adding up all of the edge weights in the spanning tree) compared to any other spanning trees.

For example, consider the following graph:



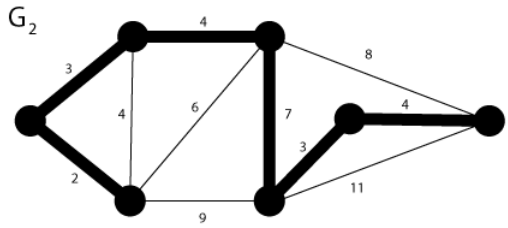
1. Find and highlight two examples of a tree in the graph.
2. Find and highlight two spanning trees in the graph.

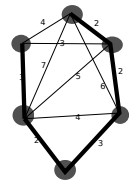
To find a minimum spanning tree for a graph, we use the following algorithm:

Prim's Algorithm

1. Select a starting vertex. It is now in the tree.
2. For any vertices in the tree, select the neighbour incident to the edge with the smallest weight and add it to the tree.
3. Continue until all vertices are in the tree. This is the minimum spanning tree!

Using Prim's Algorithm, find a minimum spanning tree from the following graphs:

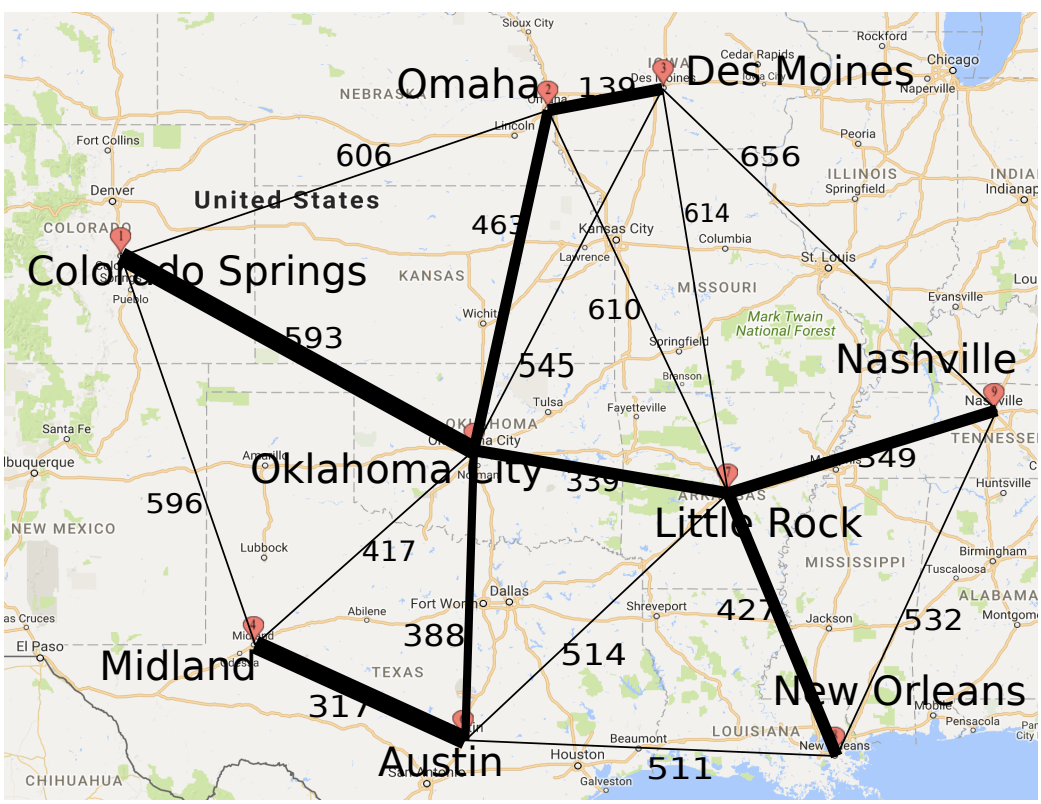




Now, consider the Traveling Salesman Problem (from the warm-up).

There is no algorithm to solve the problem - if you find one you could be a millionaire!

If we consider a variation on the problem as a minimum spanning tree problem, we could use Prim's algorithm to solve it (we can't return to the starting spot with a tree though!):



Problems

1. For the following graphs,

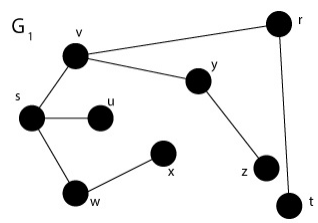
a) list the edges and vertices of the graphs

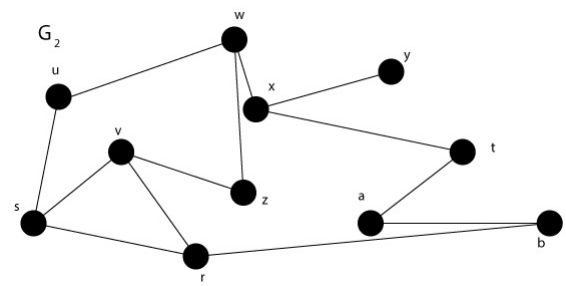
b) list the neighbours of w

c) How many edges are incident with s?

d) Find a walk between s and t. Is your walk a path?

e) Is the graph a tree? If not, find and highlight a cycle.





G1

a) Vertices: r,s,t,u,v,w,x,y,z

Edges: tr, rv, vy, yz, vs, su, sw, wx

b) s, x

c) 3

d) It is a path.

e) It is a tree.

G2

a) Vertices: a,b,r,s,t,u,v,w,x,y,z

Edges: ab,br,rs,sv,rv,vz,zw,su,uw,,wx,xy,xt,ta

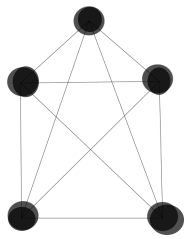
b) u,z,x

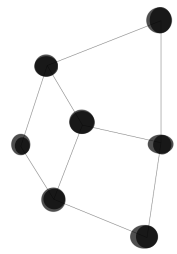
c) 3

d) It is a path

e) It is not a tree: cycle between r,s,v

2. Are the following graphs planar? Why or why not?





1 - No: 10 edges, 5 vertices

2 - No: Can't draw it with edges not crossing

3 - Yes: edges do not cross

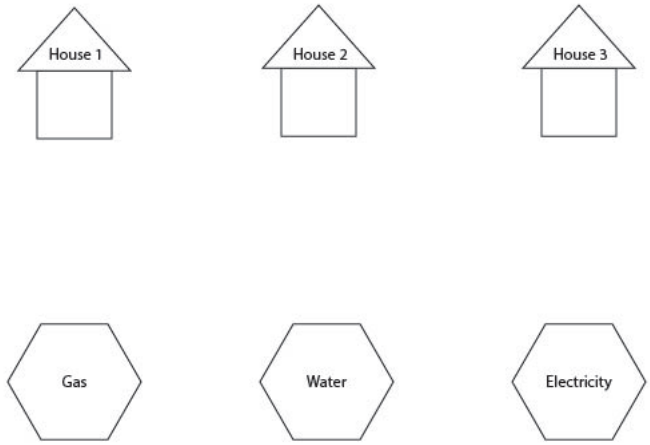
3. If a graph has 4 edges and 3 vertices, could it be a planar graph?

It could not be planar: $4 > 3(3)$-6

4. If a graph has 3 edges and 4 vertices, could it be a planar graph?

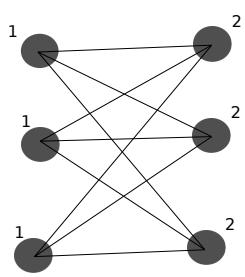
It could be planar, we don't know for sure: $3 < 3(4)$ - 6

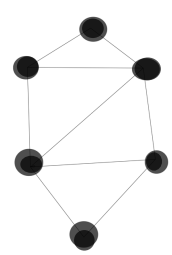
5. Each of House 1, House 2 and House 3 needs to get water, gas and electricity. The challenge is for you to connect each house to each utility. The catch: No lines can cross! Is it possible?

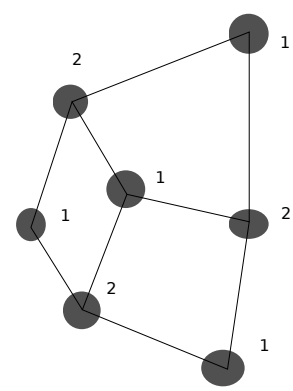


It is not possible: This becomes the K_3, 3 graph, which is not planar despite satisfying the inequality. It cannot be drawn in a way such that no edges cross.

6. Are the following graphs bipartite? If so, label the vertices to indicate the bipartition.

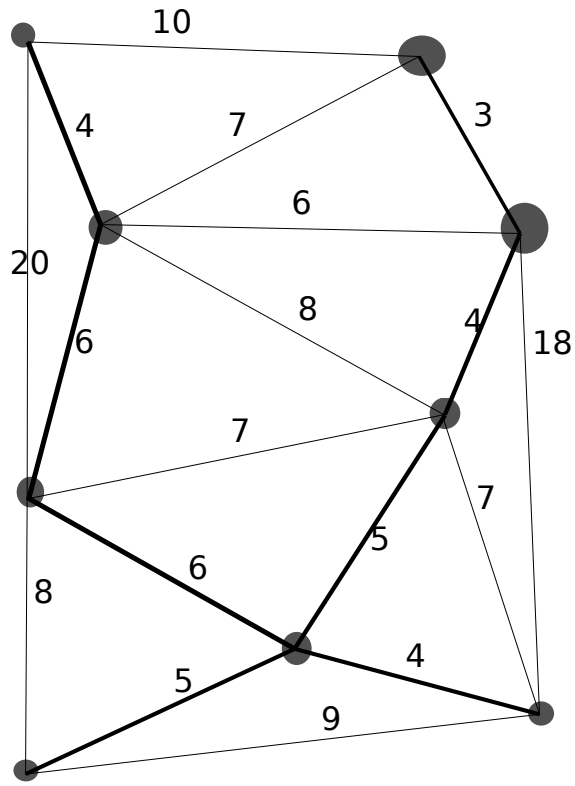




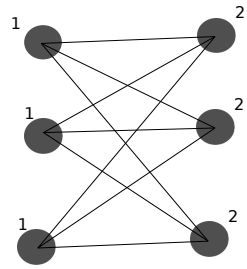


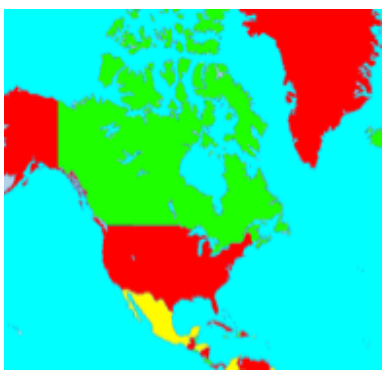
The second graph is not bipartite.

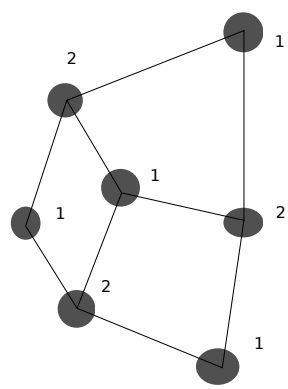
7. Evan is building the new high speed train route around the Waterloo area. He has a map with the stops and the distance between each stop. Where should Evan build the tracks to minimize the cost of construction yet ensure one can get between each stop on the train?



8. Colour the following graphs or maps with as few colours as possible. State what this number is below the map.



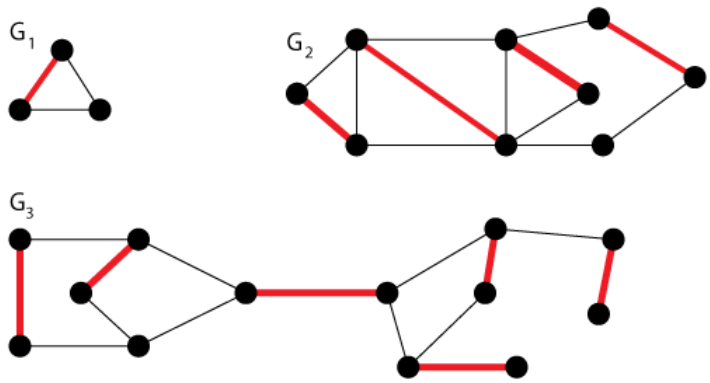




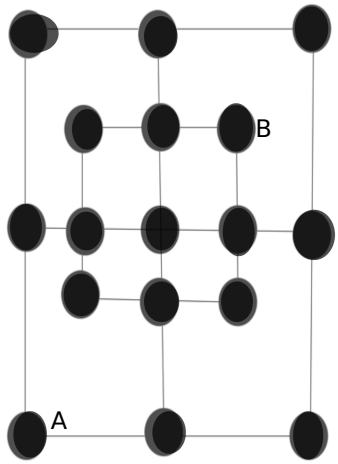
Maps of South America and North America retrieved from: $http://jaxknits.blogspot.ca/2008/12/$ and

$https://upload.wikimedia.org/wikipedia/commons/a/a9/World_map_colored_using_the_four_color_theorem_including_oceans.png$ respectively

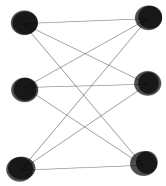
9. A **matching** of a graph is a set of edges where no edges in the set share an endpoint. The size of a matching is the number of edges in the set.
Find the maximum matching for the following graphs:



10. For the following graph:
a) Give an example of a walk from A to B using repeated vertices.
b) Give an example of a path from A to B.
c) Give an example of a cycle.
a) A walk can be defined as any connection of adjacent vertices.
b) A path can be defined as any connection of adjacent vertices so long as no vertices are repeated.
c) A cycle can be defined by the rectangle in the centre.



11. The **degree** of a vertex is the number of edges incident to it.
The **Handshaking Lemma** states that for any graph, the sum of all degrees in the graph (for all vertices) is equal to two times the number of edges in the graph.
For example, consider the following graph:



Each vertex has 3 degrees and there are 6 vertices, so the sum of degrees is $3 \times 6 = 18$.
There are 9 edges in total.
Therefore, the sum of degrees is equal to twice the number of edges since $9 \times 2 = 18$.
Prove the Handshaking Lemma.
Each edge is attached to two vertices. Each edge contributes 1 degree to an incident vertex.
So the sum of all degrees (for all vertices) is 2 times the number of edges.