UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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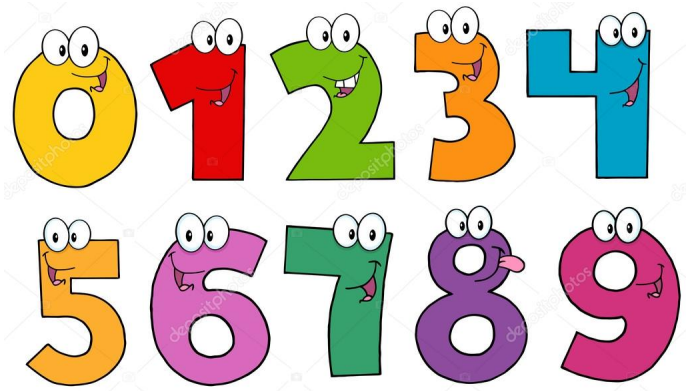
Number Theory

Warm-up!

What is special about the following groups of numbers?

3,4,5 5,12,13 7,24,25

Today we'll look at the relationship between numbers - known as **number theory**.



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Prime Numbers

A **prime** number is an integer greater than 1 whose only positive factors are 1 and itself.

A **composite** number is a number that has more positive factors than just 1 and itself.

Note: 0 and 1 are neither prime nor composite numbers.

For example, 2, 3, and 5 are prime numbers, since $2 = 2 \times 1$ only.

4 is a composite number since $4 = 2 \times 2$ and $4 = 4 \times 1$.

One method for finding the prime numbers is by using the **Sieve of Eratosthenes**.
Here are the steps to this algorithm, using the following table:
1. Cross out 1 (it is not prime)
2. Circle 2 (it is prime) and then cross out all multiples of 2
3. Circle 3 (it is prime) and then cross out all multiples of 3
4. Circle 5, then cross out all multiples of 5
5. Circle 7, then cross out all multiples of 7
6. Continue by circling the next number not crossed out, then cross out all of its multiples

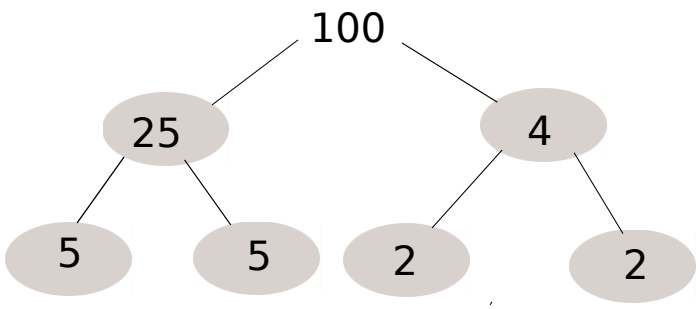
The circled numbers are all the prime numbers less than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime Factorization

Any integer greater than 1 can be expressed as the product of prime numbers.
For example, $100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$
To find a number's prime factorization, find any of its factors.
Then, break the factors down until they are prime numbers.
For 100, we know that $100 = 25 \times 4$.
$25 = 5 \times 5$
$4 = 2 \times 2$
2 and 5 are prime numbers, so we are done.
An easier way to perform prime factorization on a number is by using a **factor tree**, to break down the factors of a number into primes.

Here is a factor tree of 100:



Find the following prime factorizations:

a) 21
b) 64
c) 75
d) 97
e) 478
f) 816

Greatest Common Divisor

The **greatest common divisor** (GCD) of two numbers is the greatest number that is a factor of both numbers. For example, the GCD of 24 and 15 is 3.

<u>How to find the GCD of two numbers:</u>

1. Find the prime factorizations of both numbers
2. Multiply the common prime factors together
3. The product is the GCD

For example, let's find the GCD of 24 and 36.

$24 = 2 \times 2 \times 2 \times 3$

$36 = 2 \times 2 \times 3 \times 3$.

The prime factorizations of 24 and 36 have the following in common: 3,2,2

So, the GCD of 24 and 36 is $2 \times 2 \times 3 = 12$.

GCD(24,36)=12

1. Find the GCD of 36 and 75.

2. Find the GCD of 105 and 84.

Least Common Multiple

We know that a **multiple** of a number is that number multiplied by any whole number.

A **common multiple** of two numbers is a number that is a multiple of both numbers.

The **least common multiple** (LCM) is the smallest number that is a multiple of both numbers.

<u>How to find the LCM of two numbers</u>

1. Find the prime factorizations of both numbers
2. Multiply the prime factors the greatest number of times they appear in either number
3. The product is the LCM

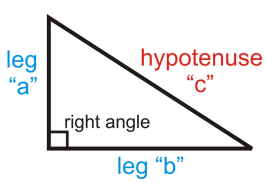
For example, let's find the LCM of 9 and 12.
$9 = 3 \times 3$
$12 = 2 \times 2 \times 3$.
3 appears twice in 9, which is greater than the one time it appears in 12.
2 appears twice in 12.
So, the LCM of 9 and 12 is $3 \times 3 \times 2 \times 2 = 36$.
LCM(9,12)=36

1. Find the LCM of 7 and 15.

2. Find the LCM of 8 and 14.

Pythagorean Triples

The **Pythagorean Theorem** says that for a right triangle with arm lengths a and b, the hypotenuse c has the following property: $a^2 + b^2 = c^2$



Retrieved from: https://www.ck12.org/geometry/the-pythagorean-theorem/

A **Pythagorean triple** is a set of three numbers that satisfy the Pythagorean theorem.
Do you know any Pythagorean triples? What do you notice about these numbers?
(Hint: check the warm-up!)

Pythagorean Triples Results

a) If a is prime, then b and c differ by 1 (harder proof: don't worry if you can't follow!)

We can rewrite $a^2 + b^2 = c^2$ as $a^2 = c^2 - b^2$

Since a is prime, we write a^2 as either $a \times a$ or as $a^2 \times 1$ (let's choose this one!)

We can rewrite $c^2 - b^2$ as $(c - b)(c + b)$

Then $a^2 \times 1 = (c - b)(c + b)$

Let $a^2 = (c + b)$ and $1 = (c - b)$

Since $c - b = 1$, we have that b and c differ by 1!

b) Multiples of Pythagorean triples are also Pythagorean triples

$3^2 + 4^2 = 5^2$

$(3 \times 2)^2 + (4 \times 2)^2 =$

$(3 \times 3)^2 + (4 \times 3)^2 =$

In general,

$(3 \times \_\_)^2 + (4 \times \_\_)^2 =$

c) There is an infinite number of Pythagorean triples (follows from part (b)!)

Finding Pythagorean Triples

Method 1: Using any odd number a

Let a be an odd number.

To find b and c, we take $\frac{a^2}{2}$.

Since b and c differ by 1 (for a prime), we will take the numbers above and below $\frac{a^2}{2}$ to be b and c!

Find Pythagorean triples containing the following numbers:

a) 5 b) 13 c) 37

Method 2: Using Multiples of Pythagorean Triples

Multiply a, b and c by the same whole number to get another Pythagorean triple.

Find Pythagorean triples containing the following numbers:

a) 6 b) 10 c) 9 (using multiples!)

Palindromic Numbers

What do you notice about the following words and numbers?

HANNAH 101 RACECAR 7777

A **palindrome** is a word that reads the same from left to right.

A **palindromic number** is a number that reads the same from left to right.

<u>Palindrome Riddles!</u>

1. What is a 6-digit palindromic number adding to 6?

2. What is the smallest 4-digit palindromic number?

3. What palindromic number has a sum and product of its digits equal to 8?

<u>One Way to Generate Palindromic Numbers</u>

1. Pick any number
2. Reverse the digits of the number
3. Add these two numbers together

4. Repeat until you get a palindromic number

Use the following numbers to create palindromic numbers! How do you know they are palindromic numbers?

28 336

Problems

1. How many 2-digit palindromes are there?

2. Which of the following are palindromes?
a) HANNAHBANANABHANNAH
b) 1030201
c) ABCDECBA
d) 17771

3. Find a palindromic number from the following numbers:
a) 127
b) 987
c) 45

4. 2002 was the last palindromic year. What will be the next palindromic year?

5. Santa forgot the pin number to unlock his sleigh. He did remember that it was the next palindromic number after 5678. What is Santa's pin number?

6. Evan's favourite palindromic number is even with the sum of its digits equal to 4. What is Evan's favourite number?

7. How many palindromic numbers are there between the numbers 100 and 400?

8. How many 7-digit palindromic numbers are there?

9. How many 8-digit palindromic numbers are there?

10. Find the prime factorization of the following numbers:
a) 472
b) 87
c) 10500 (you might need a calculator here)
d) 360

11. Find the GCD of the following pairs (or groups) of numbers:
a) 37 and 73

b) 50 and 200
c) 172 and 360
d) 328, 216 and 400

12. Find the LCM of the following pairs (or groups) of numbers:
a) 13 and 39
b) 42 and 24
c) 12 and 14
d) 15, 21 and 18

13. Do the following triples satisfy the Pythagorean theorem?
a) 9, 23, 24
b) 12, 35, 37
c) 39, 760, 761
d) 52, 170, 171

14. Find Pythagorean triples containing the following numbers.
a) 7
b) 13
c) 8
d) 25

15. The prime numbers between 10 and 20 are added together to form the number Q. What is the largest prime divisor of Q?

16. A two-digit number is divisible by 8, 12 and 18. What is the number?

17. A *perfect square* is the number that you get when you square another number. Some examples of perfect squares are 1, 4, 9, 16 and 25. What is the LCM of 14 and 18 that is also a perfect square?