Intermediate Math Circles
Fall 2018
Patterns & Counting

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Opening Problem

What are the next three terms in the sequence $1, 2, 3, 4, \ldots$?

Does everyone understand what "$\ldots$" means?

What is a Pattern?

In general, what is a pattern?

The *Merriam-Webster Dictionary* has more than a few definitions of a pattern (`https://www.merriam-webster.com/dictionary/pattern`).

Here is the definition I am going with

Pattern

A natural or chance configuration.

What is a configuration according to the *Merriam-Webster Dictionary*?

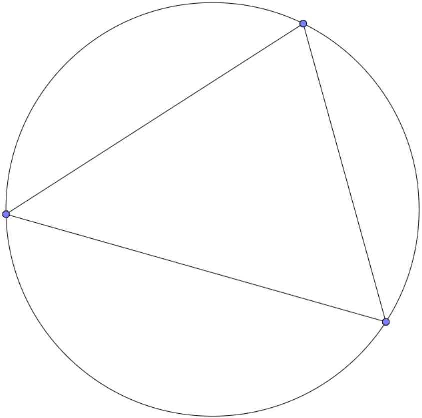
Configuration

A relative arrangement of parts or elements.

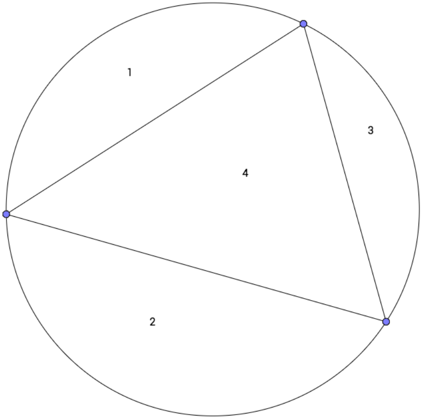
Circle Chords and Regions

Place n distinct points around a circle in such a way that no three chords share a common point in the interior. Draw all the chords. Determine the number of regions all these chords divides the circle into.

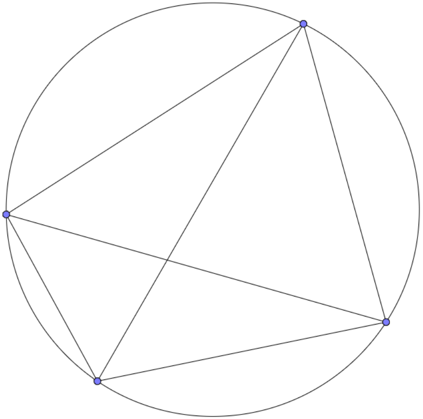
The $n = 3$ Case



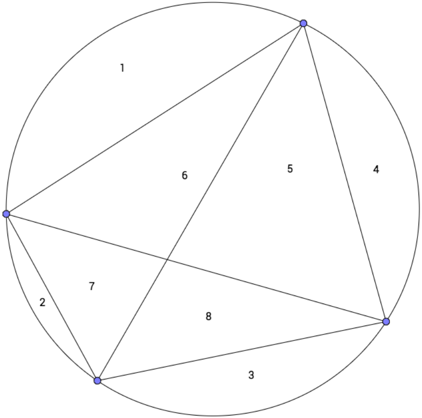
The $n = 3$ Case



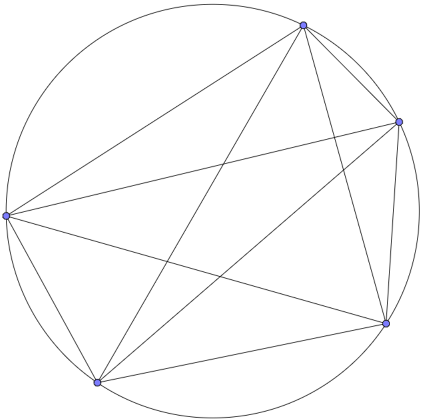
The $n = 4$ Case



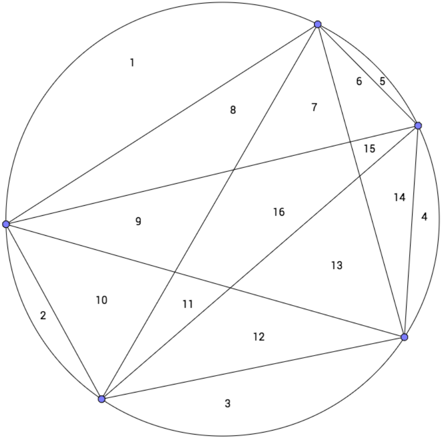
The $n = 4$ Case



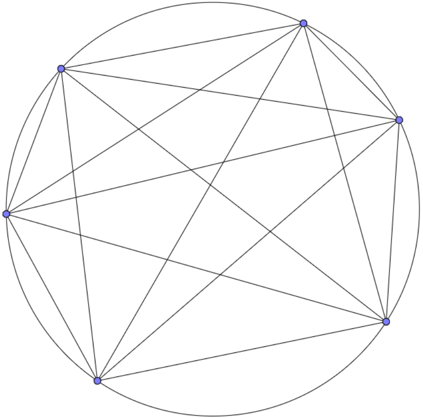
The $n = 5$ Case



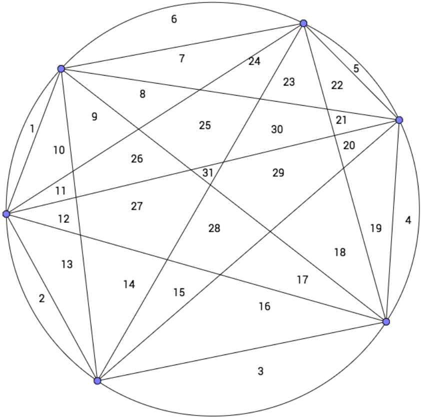
The $n = 5$ Case



The $n = 6$ Case



The $n = 6$ Case


6
7
24
23
22
5
8
9
25
30
21
1
10
26
20
31
29
11
27
12
28
19
4
18
13
14
17
15
2
16
3

Someone's Triangle

The first five rows of a number pattern are shown. If the pattern is continued

(a) What will the numbers be in the next row?

(b) What will the second number be in the fifteenth row?

(c) What will the third number be in the eighth row?

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Triangular Numbers

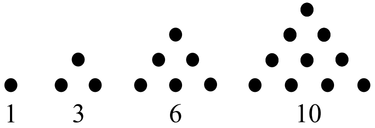
The *triangular number*, T_n, is a number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one.

The first few triangular numbers are $1, 3, 6, 10, 15,$ and 21.

`http://mathworld.wolfram.com/TriangularNumber.html`

Y2K

The first four triangular numbers $1, 3, 6,$ and $, 10$ are illustrated on the right. What is the tenth triangular number?



Formula for Triangular Numbers

What's the formula for generating triangular numbers?

Hint: Rectangles are nicer than triangles.

Square Numbers

A *square number*, also called a perfect square, is a number of the form $S_n = n^2$, where n is a positive integer. The first few square numbers are $1, 4, 9, 16, 25, 36,$ and $, 49.$

`http://mathworld.wolfram.com/SquareNumber.html`

Sum of Squares

What's the formula for the sum of square numbers?

That is, what is the sum of

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 \text{ ?}$$

Do you remember our hint from earlier? Let's change it a little and try to use the same idea to develop a formula for the expression above.

Puzzle Activity

Consider the arrangement on the left representing the sum $1^2 + 2^2 + 3^3$.

We are going to shift the cubes to help us develop the formula as shown in the arrangement on the right.

1.) Form groups of five or six. Each group needs to have three pyramid pieces as shown below.

2.) **Without breaking the pieces**, try and arrange the three pieces to form a rectangular prism. What is the closest arrangement you can get to a rectangular prism?

3.) Confirm with the instructor that you have the correct arrangement.

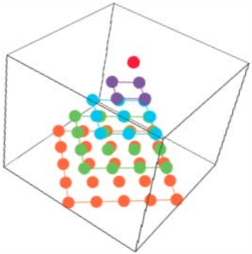
4.) Imagine your each piece represented $1^2 + 2^2 + 3^2 + \cdots + n^2$ instead of $1^2 + 2^2 + 3^2$. Using the arrangement you have found determine the number of cubes in your rectangular-ish prism if your pieces represented $1^2 + 2^2 + 3^2 + \cdots + n^2$.

5.) Use the information in four to derive a formula for $1^2 + 2^2 + 3^2 + \cdots + n^2$.

Square Pyramidal Numbers

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{(n)(n+1)(2n+1)}{6}$$

The numbers generated by this formula are called *square pyramidal numbers* and correspond to a configuration of points which form a square pyramid. The first few are $1, 5, 14, 30, 55, 91, 140,$ and 204



http://mathworld.wolfram.com/SquarePyramidalNumber.html

Problem Time

Please try the problems in *Triangular Numbers* and *Patterns* sections of the problem set.

Quick Survey

How many of you attended math circles last fall?

How many of you remember or attended the talk about counting techniques?

How many of you know something about permutations and combinations?

A Crash Course in Counting

Let's start with some definitions.

Combinatorics

The study of the method of counting.

Permutation

An arrangement of a set of objects where the order is important.

Example: PIN for a debit or credit card

Combination

A selection from a group of objects. The selection being an unordered arrangement of objects.

Examples: Picking players for a soccer team or Lotto 649

Vacation Outfits

Mike is on vacation and wants to get dressed for the day. He has 2 different hats, 3 different bottoms, and 2 different tops. How many different outfits could he have?

Solution:

Let J and E represents the two hats.
Let K, C, and U represent the three bottoms.
Let S and T represent the two tops.

What if I am NOT on vacation?

How do you approach the problem if I have 21 hats, 33 tops and 12 bottoms?

Tool # 1

Tool # 1: The Product Rule (a.k.a. The Fundamental Counting Principle)

If a first action can be performed in p ways AND a second action can be performed in q ways, then the two actions can be performed together in $p \times q$ ways.

Vacation Outfits

Mike is on vacation and wants to get dressed for the day. He has 2 different hats, 3 different bottoms, and 2 different tops. How many different outfits could he have?

Just Keep Swimming

Mike, Jen, Carmen, John, and Fiona go to the movie theatre to see Finding Dory. They all sit together in the same row with 5 seats. How many seating arrangements are possible?

Tool # 2

Writing out $5 \times 4 \times 3 \times 2 \times 1$ isn't bad, but what happens if we have 25 people?

We would have to write out

$$25 \times 24 \times 23 \times 22 \times \cdots \times 4 \times 3 \times 2 \times 1$$

Tool # 2: Factorial Notation

The product of the first n natural numbers is called n factorial and is written $n!$.
In other words

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Note: $0! = 1$

Practice!

Simplify

(a) $\dfrac{9!}{7!}$ (b) $\dfrac{8!}{5!3!}$ (c) $\dfrac{(n+3)!}{(n+1)!}$

Mike, Jen, Carmen, John, and Fiona have now decided to go tobogganing. They decide to race each other down the hill. How many different top three finishes are possible?

Not Enough Room

Consider n distinct objects and we want to know how many arrangements of length r exist.

We are assuming $n \geq r$ and both n and r are natural numbers.

Notations: ${}_nP_r \quad P(n, r) \quad n^{(r)}$

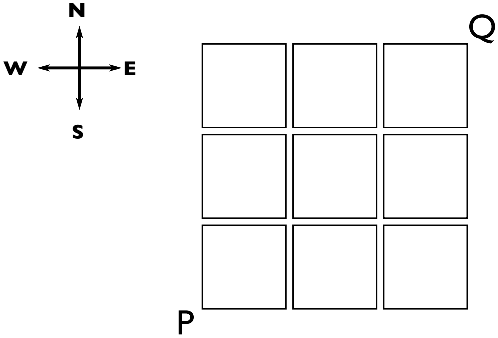
Tool # 3

Tool # 3: The Sum Rule (a.k.a. The Addition Principle)

If two actions cannot occur at the same time (are mutually exclusive), and one can be done in m ways and the other can be done in n ways, then there are $m + n$ ways in which the first OR second action can be performed.

New Kids On The Block

Intersection Q is three blocks east and three blocks north of intersection P. A person walks only North or East from P to Q along the streets shown. How many paths are there from P to Q?



Anagrams

An **anagram** is a word, phrase or sentence formed from another by rearranging its letters.

Examples:

apple inc $\Longrightarrow$ epic plan

justin timberlake $\Longrightarrow$ i am a jerk but listen

tom cruise $\Longrightarrow$ i am so cuter

Identical Objects

(a) How many permutations are there of the letters in the word LOL?
(b) How many permutations are there of the letters in the word ABBA?
(c) How many permutations are there of the letters in the word PEPPERS?

Tool # 4

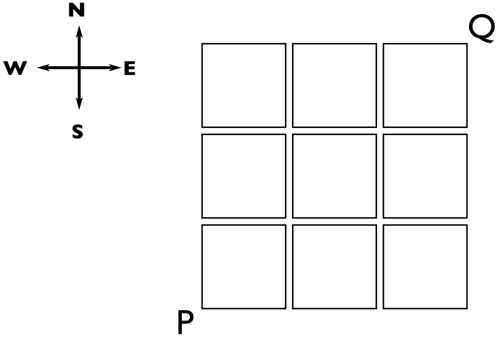
Tool # 4: Permutations with Identical Objects

The number of permutations of n objects, of which a objects are alike, another b objects are alike, another c objects are alike, and so on is:

$$\frac{n!}{a!b!c!\cdots}$$

Reunion of New Kids On The Block

Intersection Q is three blocks east and three blocks north of intersection P. A person walks only North or East from P to Q along the streets shown. How many paths are there from P to Q?



Ready, Set, Dodgeball!

Five people join a dodgeball team. Each person “friends” each of the other people on Facebook. What is the total number of friendships?

On the Road Again

I have five friends I could to invite on a road trip to Detroit. The only problem is my car only has three available seats. How many different car groups are possible?

Tool # 5

Tool # 5: Combinations

Remember a combination is an <u>unordered</u> arrangements of objects.

A combination of n different objects taken r at a time is

Notations: ${}_nC_r$ $\quad C(n, r)$ $\quad \binom{n}{r}$

Classic School Visit Question

(a) How squares are in the 4 by 4 grid below?

(b) How rectangles are in the 4 by 4 grid below?