Problem of the Week

Problem D and Solution

A Three-peat

Problem

Three tennis balls numbered 1, 2, and 3 are placed in a bag. A ball is drawn from the bag and the number is recorded. The ball is returned to the bag. After this is done three times, what is the probability that the sum of the three recorded numbers is less than 8?

Solution

In order to determine the probability, we must determine the number of ways three balls whose sum is less than 8 can be drawn from the bag and divide by the total number of ways three balls can be drawn from the bag.

First, let's determine the total number of ways three balls can be drawn from the bag. The balls are replaced after each draw so each time a ball is drawn from the bag it could be a 1, 2 or 3. Since three draws are made and there are three possible outcomes per draw, there are $3 \times 3 \times 3 = 27$ possible ways to draw three balls from the bag.

In Solution 1, we take a direct approach to counting the number of ways a sum of less than 8 can be obtained. In Solution 2, our approach is indirect. We count the number of ways a sum of 8 or more can be obtained and subtract this number from 27 to obtain the desired sum. In this problem it is actually easier to count the desired sum in this indirect way.

Solution 1

Let's determine how many of the 27 draws result in a sum that is less than 8 by systematically looking at the possible selections.

- Ball 1 is drawn three times. In this case the sum will be $1 + 1 + 1 = 3 < 8$. This can be done only 1 way: $1, 1, 1$.
- Ball 1 is drawn twice and ball 2 is drawn once. In this case the sum will be $1 + 1 + 2 = 4 < 8$. This can be done 3 ways: $1, 1, 2$ or $1, 2, 1$ or $2, 1, 1$.
- Ball 1 is drawn twice and ball 3 is drawn once. In this case the sum will be $1 + 1 + 3 = 5 < 8$. This can be done 3 ways: $1, 1, 3$ or $1, 3, 1$ or $3, 1, 1$.
- Ball 1 is drawn once and ball 2 is drawn twice. In this case the sum will be $1 + 2 + 2 = 5 < 8$. This can be done 3 ways: $1, 2, 2$ or $2, 1, 2$ or $2, 2, 1$.
- Ball 1 is drawn once and ball 3 is drawn twice. In this case the sum will be $1 + 3 + 3 = 7 < 8$. This can be done 3 ways: $1, 3, 3$ or $3, 1, 3$ or $3, 3, 1$.

- Ball 1 is drawn once, ball 2 is drawn once and ball 3 is drawn once. In this case the sum will be $1 + 2 + 3 = 6 < 8$. This can be done 6 ways: $1, 2, 3$ or $1, 3, 2$ or $2, 1, 3$ or $2, 3, 1$ or $3, 1, 2$ or $3, 2, 1$.

- Ball 2 is drawn three times. In this case the sum will be $2 + 2 + 2 = 6 < 8$. This can be done only 1 way: $2, 2, 2$.

- Ball 2 is drawn twice and ball 3 is drawn once. In this case the sum will be $2 + 2 + 3 = 7 < 8$. This can be done 3 ways: $2, 2, 3$ or $2, 3, 2$ or $3, 2, 2$.

- Ball 2 is drawn once and ball 3 is drawn twice. In this case the sum will be $2 + 3 + 3 = 8$, which is not less than 8.

- Ball 3 is drawn three times. In this case the sum will be $3 + 3 + 3 = 9$, which is not less than 8.

We see that there are $1 + 3 + 3 + 3 + 6 + 3 + 1 + 3 = 23$ ways to draw the balls so that the sum of the recorded numbers is less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$.

Solution 2

Let's determine how many of the 27 draws result in a sum that is 8 or more. Since the maximum sum is 9, we need to count the number of ways the sum is 8 or 9.

- The sum is 8. The only way to do this is to draw ball 2 once and ball 3 twice. This can be done 3 ways: $2, 3, 3$ or $3, 2, 3$ or $3, 3, 2$.

- The sum is 9. The only way to do this is to draw ball 3 three times.

We see that there are $3 + 1 = 4$ ways to draw the balls so that the recorded sum is 8 or 9. Therefore, of the 27 outcomes, $27 - 4 = 23$ give a sum less than 8.

Therefore, the probability that the sum is less than 8 is $\frac{23}{27}$.

The indirect approach used in the second solution is definitely more efficient!

For Further Thought:

If this were a game, it would be $unfair$ since the probability of obtaining a sum less than 8 is $\frac{23}{27}$ or 85% while the probability of obtaining a sum of 8 or higher is $\frac{4}{27}$ or 15%. In a fair game, we want the chance of two different outcomes occurring to be the same. Can you create a fair game out of this problem?