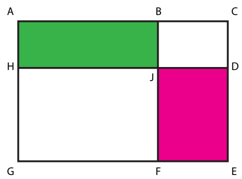


Problem of the Week

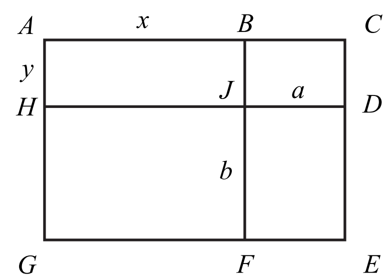
Problem D and Solution

Maximize the Area

Problem

Two rectangles, $ABJH$ and $JDEF$, with integer side lengths, share a common corner at J such that HJD and BJF are perpendicular line segments. The two rectangles are enclosed by a larger rectangle $ACEG$, as shown. The area of rectangle $ABJH$ is 6 cm^2 and the area of rectangle $JDEF$ is 15 cm^2. Determine the largest possible area of the rectangle $ACEG$. Note that the diagram is not intended to be to scale.

Solution



Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.

Therefore,
$$\begin{aligned} AB &= HJ = GF = x, \\ AH &= BJ = CD = y, \\ BC &= JD = FE = a, \text{ and} \\ HG &= JF = DE = b. \end{aligned}$$

$$\begin{aligned} \text{Then area}(ACEG) &= \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG) \\ &= 6 + ya + 15 + xb \\ &= 21 + ya + xb \end{aligned}$$

Since the area of rectangle $ABJH$ is 6 cm^2 and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1$ cm and $y = 6$ cm, $x = 6$ cm and $y = 1$ cm, $x = 2$ cm and $y = 3$ cm, or $x = 3$ cm and $y = 2$ cm.

Since the area of rectangle $JDEF$ is 15 cm^2 and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1$ cm and $b = 15$ cm, $a = 15$ cm and $b = 1$ cm, $a = 3$ cm and $b = 5$ cm, or $a = 5$ cm and $b = 3$ cm.

To maximize the area, we need to pick the values of x, y, a, b which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1$ cm and $y = 6$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 6$ cm and $y = 1$ cm with the four possibilities of a and b. Similarly, trying $x = 2$ cm and $y = 3$ cm with the four possibilities of a and b will give the same 4 areas, in some order, as trying $x = 3$ cm and $y = 2$ cm with the four possibilities of a and b. (As an extension, we will leave it to you to think about why this is the case.)

Case 1: $x = 1$ cm, $y = 6$ cm and $a = 1$ cm, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42 \text{ cm}^2$$

Case 2: $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112 \text{ cm}^2$$

Case 3: $x = 1$ cm, $y = 6$ cm and $a = 3$ cm, $b = 5$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44 \text{ cm}^2$$

Case 4: $x = 1$ cm, $y = 6$ cm and $a = 5$ cm, $b = 3$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 6(5) + 1(3) = 54 \text{ cm}^2$$

Case 5: $x = 2$ cm, $y = 3$ cm and $a = 1$, $b = 15$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54 \text{ cm}^2$$

Case 6: $x = 2$ cm, $y = 3$ cm and $a = 15$, $b = 1$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(15) + 2(1) = 68 \text{ cm}^2$$

Case 7: $x = 2$ cm, $y = 3$ cm and $a = 3$, $b = 5$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40 \text{ cm}^2$$

Case 8: $x = 2$ cm, $y = 3$ cm and $a = 5$, $b = 3$ cm

$$\text{area}(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42 \text{ cm}^2$$

We see that the maximum area is 112 cm^2, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm^2.