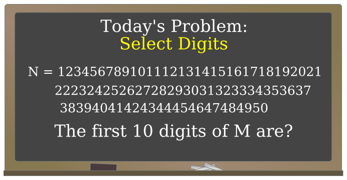

Problem of the Week
Problem E and Solution
Select Digits

Problem

An integer, N, is formed by writing the integers from 1 to 50 in order. That is,
$N =$1234567891011121314151617181920212223242526272829303132333435363738394041424344454647484950.

We will abbreviate N by writing only the first twelve positive integers and the last three integers, 48, 49, and 50. We place three dots between 12 and 48 to represent all of the integers between 12 and 48. So $N = 123456789101112\ldots484950$. Some of the digits are selected from N and discarded. The remaining digits, in their original order, form a new integer such that the sum of the digits of this new number is 200. If M is the largest number that can be formed this way, what are the first ten digits of M?

Solution

We start by determining the sum of the digits of N. This is the same as determining the sum of the digits of the numbers from 1 to 50.

The digits 1 to 9 sum to $1+2+3+4+5+6+7+8+9=45$. Note also that the digits 0 to 9 sum to 45.

The digits in the numbers 10 to 19 sum to
$(1+0)+(1+1)+(1+2)+(1+3)+(1+4)+(1+5)+(1+6)+(1+7)+(1+8)+(1+9)$
$=10(1)+(0+1+2+3+4+5+6+7+8+9)=10+45=55$.

The digits in the numbers 20 to 29 sum to
$(2+0)+(2+1)+(2+2)+(2+3)+(2+4)+(2+5)+(2+6)+(2+7)+(2+8)+(2+9)$
$=10(2)+(0+1+2+3+4+5+6+7+8+9)=20+45=65$.

Similarly, the digits in the numbers 30 to 39 sum to $10(3)+45=75$ and the digits in the numbers 40 to 49 sum to $10(4)+45=85$.

We must add 5 and 0 in order to account for the number 50 at the end of N. Therefore, the sum of the digits of N is

$$45+55+65+75+85+(5+0)=330.$$

Since the digits of M sum to 200, the digits that are removed and discarded must sum to $330-200=130$.

In order for M to be as large as possible, we need M to have as many digits as possible. So we need to remove as few digits as possible such that the digits that are removed sum to 130. We can do so by removing the largest digits first.

We start by removing the 9's. There are five 9's in N (from 9, 19, 29, 39 and 49). These five 9's sum to $5 \times 9 = 45$. After removing all of the 9's, we still need to remove digits that sum to $130 - 45 = 85$.

We now remove the 8's. There are five 8's in N (from 8, 18, 28, 38 and 48). These five 8's sum to $5 \times 8 = 40$. After removing all of the 8's, we still need to remove digits that sum to $85 - 40 = 45$.

We now remove the 7's. There are five 7's in N (from 7, 17, 27, 37 and 47). These five 7's sum to $5 \times 7 = 35$. After removing all of the 7's, we still need to remove digits that sum to $45 - 35 = 10$.

The remaining digits are

$$1234561011121314151611120212223242526222303132333435363334041424344454644450.$$

We are now left with only the digits 0, 1, 2, 3, 4, 5 and 6. We need to remove as few of these digits as possible that sum to 10. We cannot do so by removing only one digit, but can remove two digits that sum to 10 if we remove a 6 and 4 or two 5's. Either way M will have the same number of digits, regardless of if we remove a 6 and a 4 or two 5's.

So which digits do you remove and where do you remove them from?

The remaining digits start $123456101112\cdots$.

Removing a 6 and a 4 or a 5 and a 5 from anywhere after the first six digits will result in a number whose first six digits are 123456.

Removing the 6 from the beginning of the number and a 4 from anywhere past this 6 will result in a number whose first 6 digits are 123451.

Removing the 5 from the beginning of the number and the other 5 from anywhere else in the number will result in a number whose first 6 digits are 123461.

Removing the 4 from the beginning of the number and a 6 from anywhere else in the number will result in a number whose first 6 digits are 123510 or 123561.

Since $123561 > 123510 > 123461 > 123456 > 123451$, removing the first 4 and not the first 6 will lead to the largest value of M.

After removing the first 4, we are left with

$$12356101112131415161112021222324252622230313233343536333404142434445\underset{\Uparrow}{6}44450.$$

We still must remove a 6 to bring our digit sum to 200. We want whatever 6 we remove to affect the size of the final number in the least possible way. We need to therefore remove the 6 with the lowest place value. The 6 to be removed is the hundred thousands digit in the number shown just above. After removing the 6,

$$M = 12356101112131415161112021222324252622230313233343536333404142434445444450.$$

And the first 10 digits of M are 1235610111.