Problem of the Week
Grade 11 and 12

What are the Chances of That?
Solution

Problem

Use each of the digits one to nine exactly once to form a nine digit number. Determine the probability that the number formed is divisible by 4 or ends in 8.

Solution

In creating a 9 digit number, there are 9 choices for the first digit, 8 choices for the second digit, 7 choices for the third digit, and so on. So the number of 9 digit numbers is $9 \times 8 \times 7 \times \cdots \times 2 \times 1 = 9! = 362\ 880$.

If the nine digit number ends in 8, there is only choice for the unit's digit. There are then 8! ways to create the remaining 8 digit number. Thus, the number of 9 digit numbers ending in 8 is $1 \times 8! = 40\ 320$.

For a number to be divisible by 4, the last two digits of the number must be divisible by 4. There are 24 numbers less than 100 that are divisible by 4. This group includes 2 single digit numbers, 4 and 8, which must be excluded. It also includes 4 two digit numbers ending in zero, 20, 40, 60 and 80. Zero is not one of the nine possible digits so these numbers must be excluded. And finally the list includes 2 numbers, 44 and 88, which have repeated digits. Since each digit can be used only once, these numbers must be excluded. Therefore, there are $24 - 2 - 4 - 2$ or 16 valid two digit numbers which are divisible by 4. There are then 7! ways to create the remaining 7 digit number. So the number of 9 digit numbers divisible by 4 is $16 \times 7! = 80\ 640$.

If we added the $1 \times 8!$ and the $16 \times 7!$ we would be double counting the number of nine digit numbers which end in 8 and are divisible by 4. So we must subtract off the number of numbers which are divisible by 4 and end in 8 since they have been counted twice. There are 3 two digit numbers with distinct digits, $28, 48,$ and 68, that end in 8 and are divisible by 4. As before, there are 7! ways to create the remaining 7 digit number. So the number of 9 digit numbers with distinct digits ending in 8 and divisible by 4 is $3 \times 7! = 15\ 120$.

We can now compute the required probability as follows:

$$\frac{(\#\text{ ending in 8}) + (\#\text{ divisible by 4}) - (\#\text{ ending in 8 and divisible by 4})}{(\#\text{ of 9 digit numbers})}$$

$$= \frac{1 \times 8! + 16 \times 7! - 3 \times 7!}{362\ 880} = \frac{40\ 320 + 80\ 640 - 15\ 120}{362\ 880} = \frac{105\ 840}{362\ 880} = \frac{7}{24}.$$

$\therefore$ the probability that a nine digit number is divisible by 4 or ends in 8 is $\frac{7}{24}$.