Problem of the Week
Grade 11 and 12

Find the Number
Solution

Problem

Five different numbers, a, b, c, d, and e, are written in order smallest to largest such that $a < b < c < d < e$. When the numbers are added in pairs the following sums are obtained: $726, 756, 770, 781, 795, 816, 825, 830, 860$ and 885. Determine the sum of the five numbers and determine the value of the smallest number, a.

Solution

We know the sum of each pair so we can determine the sum of all of the sums.
$(a+b)+(a+c)+(a+d)+(a+e)+(b+c)+(b+d)+(b+e)+(c+d)+(c+e)+(d+e)$
$= 4a + 4b + 4c + 4d + 4e$
$= 726 + 756 + 770 + 781 + 795 + 816 + 825 + 830 + 860 + 885$
$= 8\ 044$
Therefore $4a + 4b + 4c + 4d + 4e = 8\ 044$. Dividing each side of the equation by 4, we obtain $a + b + c + d + e = 2\ 011$. That is, the sum of the five numbers is 2 011.

The smallest sum is created by adding the two smallest numbers. Therefore, $a + b = 726$. The largest sum is created by adding the two largest numbers. Therefore, $d + e = 885$. If we subtract the sum of the two smallest numbers and the sum of the two largest numbers from the total sum, we will get the middle number, c.

$$(a + b + c + d + e) - (a + b) - (d + e) = 2\ 011 - 726 - 885$$
$$\therefore c = 400$$

The second smallest sum is created by adding the smallest number, a, to the middle number, c. (A justification of this statement is shown on the next page.) Therefore, $a + c = 756$ and $a = 756 - c = 756 - 400 = 356$.

The sum of the five numbers is 2 011 and the smallest number is 356. The complete list is 356, 370, 400, 425 and 460. It is left as an exercise for the student to verify the correctness of this list.

Why is $a+c$ is the second smallest sum?

We know that $a<b<c<d<e$ so the following is true:

1. Since $b>a$, let $b=a+m,\ m>0$.
2. Since $c>b$, $c=b+n=a+m+n,\ m,n>0$.
3. Since $d>c$, $d=c+p=a+m+n+p,\ m,n,p>0$.
4. Since $e>d$, $e=d+q=a+m+n+p+q,\ m,n,p,q>0$.

We need to show that $a+c>a+b$ and $a+c$ is smaller than all other possible sums. So now let's look at the ten possible sums:

1. $a+b=a+a+m=2a+m$.
2. $a+c=a+a+m+n=2a+m+n>2a+m=a+b$.
   So $a+c>a+b$.
3. $a+d=a+a+m+n+p=2a+m+n+p>2a+m+n=a+c$.
   So $a+d>a+c$.
4. $a+e=a+a+m+n+p+q=2a+m+n+p+q>2a+m+n=a+c$.
   So $a+e>a+c$.
5. $b+c=a+m+a+m+n=2a+2m+n>2a+m+n=a+c$.
   So $b+c>a+c$.
6. $b+d=a+m+a+m+n+p=2a+2m+n+p>2a+m+n=a+c$.
   So $b+d>a+c$.
7. $b+e=a+m+a+m+n+p+q=2a+2m+n+p+q>2a+m+n=a+c$.
   So $b+e>a+c$.
8. $c+d=a+m+n+a+m+n+p=2a+2m+2n+p>2a+m+n=a+c$.
   So $c+d>a+c$.
9. $c+e=a+m+n+a+m+n+p+q=2a+2m+2n+p+q>2a+m+n=a+c$.
   So $c+e>a+c$.
10. $d+e=a+m+n+p+a+m+n+p+q=2a+2m+2n+2p+q>2a+m+n=a+c$.
    So $d+e>a+c$.

So we have shown that $a+c>a+b$ and $a+c$ is smaller than the eight other possible sums. Therefore, $a+c$ is the second smallest sum. (It is also possible to use the above information to show that $c+e$ is the second largest sum.)