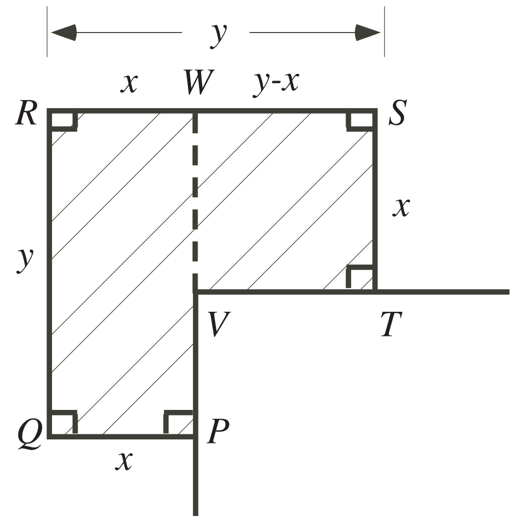
Problem of the Week
Grade 11 and 12

Playing With a Full Deck
Solution

Problem

The diagram shows the plan for a deck which is to be built on the corner of a cottage. A railing is to be constructed around the four outer edges of the deck from P to Q to R to S to T. The length of the rail from P to Q is the same as the length of the rail from S to T and the length of the rail from Q to R is the same as the length of the rail from R to S. The total length of the railing is 30 m. Determine the dimensions of the deck which will have the maximum area.



Solution

Label the corner of the cottage V. Draw a line segment through PV to RS intersecting at W. $PW \perp RS$. This makes two rectangles $PQRW$ and $WSTV$.

Let x represent the length of PQ and ST. Let y represent the length of QR and RS. Since $PQRW$ is a rectangle, $RW = PQ = x$ and $WS = RS - RW = y - x$.

Let A represent the area of the deck.

The total length of fencing is $PQ + QR + RS + ST = x + y + y + x = 2x + 2y = 30$. By dividing by 2, $x + y = 15$. Rearranging to solve for y we obtain $y = 15 - x$. (1)

$$\begin{aligned}
\text{Area of deck} &= \text{Area } PQRW + \text{Area } WSTV \\
A &= QR \times RW + WS \times ST \\
&= yx + (y - x)x \\
&= (15 - x)x + ((15 - x) - x)x, \quad \text{substituting from (1) above} \\
&= (15 - x)x + (15 - 2x)x \\
&= 15x - x^2 + 15x - 2x^2 \\
&= -3x^2 + 30x \\
&= -3(x^2 - 10x) \\
&= -3(x^2 - 10x + 5^2 - 5^2), \quad \text{by completing the square} \\
&= -3(x - 5)^2 + 75
\end{aligned}$$

This is the equation of a parabola which opens down from a vertex of $(5, 75)$. The maximum area is 75 m^2 when $x = 5$ m. When $x = 5$, $y = 15 - x = 15 - 5 = 10$ m.

The side lengths of the deck are 5 m and 10 m giving a maximum area of 75 m^2.