



# Problem of the Week

## Grade 11 and 12

### Powerful Factors at Work

### Solution

#### Problem

The product of the integers 1 to 64 can be written in an abbreviated form as  $64!$  and we say  $64$  *factorial*. So  $64! = 64 \times 63 \times 62 \times \cdots \times 3 \times 2 \times 1$ . In general, the product of the positive integers 1 to  $n$  is  $n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$ . Determine the largest positive integer value of  $n$  so that  $64!$  is divisible by  $12^n$ .

#### Solution

The prime factorization of 12 is  $2^2 \times 3$ . We must determine how many times the factors  $2^2 \times 3$  are repeated in the factorization of  $P$ .

First we will count the number of factors of 2 in  $P$  by looking at the 32 even numbers. Each of the numbers  $\{2, 4, 6, \dots, 60, 62, 64\}$  contains a factor of 2. That is a total of 32 factors of 2. Dividing the even numbers by 2, we obtain the numbers  $\{1, 2, 3, \dots, 30, 31, 32\}$ . This list contains 16 even numbers so we gain another 16 factors of 2 bringing the total to  $32 + 16 = 48$ . Dividing the even numbers by 2, we obtain the numbers  $\{1, 2, 3, \dots, 14, 15, 16\}$ . This list contains 8 even numbers so we gain another 8 factors of 2 bringing the total to  $48 + 8 = 56$ . Dividing the even numbers by 2, we obtain the numbers  $\{1, 2, 3, \dots, 6, 7, 8\}$ . This list contains 4 even numbers so we gain another 4 factors of 2 bringing the total to  $56 + 4 = 60$ . Dividing the even numbers by 2, we obtain the numbers  $\{1, 2, 3, 4\}$ . This list contains 2 even numbers so we gain another 2 factors of 2 bringing the total to  $60 + 2 = 62$ . Finally, dividing the even numbers by 2, we obtain the numbers  $\{1, 2\}$ . This list contains 1 even number so we gain another factor of 2 bringing the total to  $62 + 1 = 63$ . So when  $P$  is factored there are 63 factors of 2. In fact, the largest power of 2 that  $P$  is divisible by is  $2^{63}$ .

Next we will count the number of factors of 3 in  $P$  by looking at the 21 multiples of 3, namely  $\{3, 6, 9, \dots, 57, 60, 63\}$ . Each of these numbers 21 numbers contains a factor of 3. Dividing the numbers by 3, we obtain the numbers  $\{1, 2, 3, \dots, 19, 20, 21\}$ . This list contains 7 multiples of 3 so we gain another 7 factors of 3 bringing the total to  $21 + 7 = 28$ . Dividing the multiples of 3 by 3, we obtain the numbers  $\{1, 2, 3, 4, 5, 6, 7\}$ . This list contains 2 multiples of 3 so we gain another 2 factors of 3 bringing the total to  $28 + 2 = 30$ . Dividing the multiples of 3 by 3, we obtain the numbers  $\{1, 2\}$ . This list contains no multiples of 3 so when  $P$  is factored there is a total of 30 factors of 3. In fact, the largest power of 3 that  $P$  is divisible by is  $3^{30}$ .

$$\begin{aligned} \text{Combining the results } P \text{ is divisible by } 2^{63} \times 3^{30} &= 2^3 \times 2^{60} \times 3^{30} \\ &= 2^3 \times (2^2)^{30} \times 3^{30} \\ &= 2^3 \times ((2^2) \times 3)^{30} \\ &= 2^3 \times (12)^{30} \end{aligned}$$

$\therefore P$  is divisible by  $12^{30}$  and the largest value of  $n$  is 30. (Since all of the powers of 3 have been used, none remain to combine with any of the remaining 2's to form an additional factor of 12.)

