Problem of the Week
Grade 7 and 8

The Search is On
Solution

Problem

The digits 1, 2, 3, 4, and 5 are each used exactly once to create a five digit number $abcde$ which satisfies the following two conditions: the two digit number ab is divisible by 4; and the two digit number cd is divisible by 3. Find all five digit numbers, formed using each of the digits 1 to 5 exactly once, that satisfy both conditions.

Solution

The following two digit numbers are divisible by 4: 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, and 52. From this list, 16, 20, 28, 36, 40 and 48 can be eliminated because they use a digit not permitted in the problem. The number 44 can be removed since a digit is repeated. The list stops at 52 since all multiples of 4 beyond this point use a digit that is not permitted in the problem. So the only two digit numbers that satisfy the first condition are 12, 24, 32 and 52. Therefore the five digit number looks like $12cde$ or $24cde$ or $32cde$ or $52cde$.

For a number to be divisible by three, the sum of its digits will be divisible by three. Since cd is divisible by three, the sum $c + d$ must be divisible by three.

Let's look at the four numbers, $12cde$, $24cde$, $32cde$ and $52cde$, separately.

If the number is $12cde$, $ab = 12$ and there are only three possible digits left, 3, 4 and 5.

- If we use 3 and 4 for c and d, the sum is $c + d = 7$ which is not divisible by three.
- If we use 3 and 5 for c and d, the sum is $c + d = 8$ which is not divisible by three.
- If we use 4 and 5 for c and d, the sum is $c + d = 9$ which is divisible by three. But the order in which we use 4 and 5 does not matter for cd to be divisible by three. (The sum $c + d$ would still be 9.) So $cd = 45$ or $cd = 54$. If $ab = 12$ and $cd = 45$ then $e = 3$ and one possible solution is $abcde = 12453$. If $ab = 12$ and $cd = 54$ then $e = 3$ and another possible solution is $abcde = 12543$.

  There are two numbers of the form $12cde$ that can be made which satisfy both conditions in the problem: 12453 and 12543.

(The solution continues on the next page.)

If the number is $24cde$, $ab = 24$ and there are only three possible digits left, 1, 3 and 5.

- If we use 1 and 3 for c and d, the sum is $c + d = 4$ which is not divisible by three.
- If we use 1 and 5 for c and d, the sum is $c + d = 6$ which is divisible by three. But the order in which we use 1 and 5 does not matter for cd to be divisible by three. So $cd = 15$ or $cd = 51$. If $ab = 24$ and $cd = 15$ then $e = 3$ and one possible solution is $abcde = 24153$. If $ab = 24$ and $cd = 51$ then $e = 3$ and another possible solution is $abcde = 24513$.
- If we use 3 and 5 for c and d, the sum is $c + d = 8$ which is not divisible by three.

  There are two numbers of the form $24cde$ that can be made which satisfy both conditions in the problem: 24153 and 24513.

If the number is $32cde$, $ab = 32$ and there are only three possible digits left, 1, 4 and 5.

- If we use 1 and 4 for c and d, the sum is $c + d = 5$ which is not divisible by three.
- If we use 1 and 5 for c and d, the sum is $c + d = 6$ which is divisible by three. But the order in which we use 1 and 5 does not matter for cd to be divisible by three. So $cd = 15$ or $cd = 51$. If $ab = 32$ and $cd = 15$ then $e = 4$ and one possible solution is $abcde = 32154$. If $ab = 32$ and $cd = 51$ then $e = 4$ and another possible solution is $abcde = 32514$.
- If we use 4 and 5 for c and d, the sum is $c + d = 9$ which is divisible by three. But the order in which we use 4 and 5 does not matter for cd to be divisible by three. So $cd = 45$ or $cd = 54$. If $ab = 32$ and $cd = 45$ then $e = 1$ and one possible solution is $abcde = 32451$. If $ab = 32$ and $cd = 54$ then $e = 1$ and another possible solution is $abcde = 32541$.

  There are four numbers of the form $32cde$ that can be made which satisfy both conditions in the problem: 32154, 32514, 32451 and 32541.

If the number is $52cde$, $ab = 52$ and there are only three possible digits left, 1, 3 and 4.

- If we use 1 and 3 for c and d, the sum is $c + d = 4$ which is not divisible by three.
- If we use 1 and 4 for c and d, the sum is $c + d = 5$ which is not divisible by three.
- If we use 3 and 4 for c and d, the sum is $c + d = 7$ which is not divisible by three.

  There are no numbers of the form $52cde$ that can be made which satisfy both conditions in the problem.

Summarizing from each of the cases, there are 8 numbers, 12453, 12543, 24153, 24513, 32154, 32514, 32451 and 32541, which satisfy all of the conditions of the problem.

Note: If a student were to try to solve this problem by listing all of the possible five digit numbers, there would be 120 numbers to check.