Problem of the Week
Grade 9 and 10

Just Perfect
Solution

Problem

A *perfect square* is an integer that is the square of another integer. For example, 36 is a perfect square since $36 = 6^2$. Determine the smallest perfect square greater than 4 000 that is divisible by 392.

Solution

In order to understand the nature of perfect squares we should examine the prime factorization of a few perfect squares.

From the example, $36 = 6^2 = (2 \times 3)^2 = 2^2 \times 3^2$.
The number $144 = 12^2 = (3 \times 4)^2 = 3^2 \times (2^2)^2 = 3^2 \times 2^4$.

From the above examples we note that, for each perfect square, the exponent on each of its prime factors is even. For some integer a, if m is an even integer greater than or equal to zero then a^m is a perfect square. For example, $2^6 \times 3^4 \times 6^2 = 64 \times 81 \times 36 = 186\ 624 = 432^2$.

The number $392 = 8 \times 49 = 2^3 \times 7^2$. This clearly is not a perfect square since the exponent of 2 is odd. We require another factor of 2 to obtain a multiple of 392 that is a perfect square, namely $2 \times 392 = 784$. We now must multiply 784 by a perfect square so that the product is greater than 4 000. Since $4\ 000 \div 784 \doteq 5.1$ we must multiply 784 by the smallest perfect square greater than 5.1 to create a perfect square greater than 4 000. In this case that perfect square is 9. So the smallest perfect square greater than 4 000 that is divisible by 392 is $784 \times 9 = 7\ 056 = 84^2$. We can also show that $7\ 056 \div 392 = 18$ to verify that 7056 is divisible by 392.

$\therefore$ the smallest perfect square greater than 4 000 that is divisible by 392 is 7 056.