



Problem of the Week

Grade 9 and 10

Keep on Counting

Solution

Problem

For this problem you have been given 1000 digits made up of 100 zeros, 100 ones, 100 twos, \dots , 100 nines. You have 100 of each possible digit. Start counting by ones, from one. Each time you say a number you must remove the digits required to make the number from your stock pile of digits. What is the largest number you can count to without running out of the digits needed to form the number?

Solution

In the numbers 1 to 99, each digit, other than zero, will be used the same number of times. Zero will be used 9 less times than every other digit. As soon as you get into the 100's, the one's will be used more frequently. So, let's just count the number of ones used until we run out.

From 1 to 99, a one is used in the units digit 10 times: $\{1, 11, 21, \dots, 91\}$. There are 10 numbers with a tens digit of 1: $\{10, 11, 12, \dots, 18, 19\}$. By the time we reach 99, we have used $10 + 10 = 20$ ones.

From 100 to 109, we use 10 ones plus another in the number 101. We use 11 more ones and have now used 31 ones.

From 110 to 119 we use 10 ones for hundreds digits and 10 ones for tens digits and one 1 for the units digit in 111. We use 21 more ones and have now used 52 ones.

For 120 to 129, 130 to 139, 140 to 149 and 150 to 159 we use the same number of ones as we did counting from 100 to 109. So to get from 120 to 159 we use $11 + 11 + 11 + 11 = 44$ ones and have now used a total of 96 ones.

Now we can count until we use up the 4 remaining ones: 160, 161 and 162.

When we try to count 163, there are no ones remaining.

Therefore, the largest number we can count to before running out of the necessary digits to create the number is 162.

