Problem of the Week
Grade 9 and 10

POWERful
Solution

Problem

When $n = 7^{2011}$ is expressed as an integer, what are its last two digits?

Solution

Let's start by examining the last two digits of various powers of 7.

$$\begin{aligned} 7^1 &= \mathbf{07} \\ 7^2 &= \mathbf{49} \\ 7^3 &= 3\mathbf{43} \\ 7^4 &= 2\ 4\mathbf{01} \\ 7^5 &= 16\ 8\mathbf{07} \\ 7^6 &= 117\ 6\mathbf{49} \\ 7^7 &= 823\ 5\mathbf{43} \\ 7^8 &= 5\ 764\ 8\mathbf{01} \end{aligned}$$

Notice that the last two digits repeat every four powers of 7. The pattern continues. 7^9 ends with 07, 7^{10} ends with 49, 7^{11} ends with 43, 7^{12} ends with 01, and so on. Starting with the first power of 7, every four consecutive powers of 7 will have the last two digits 07, 49, 43, and 01.

We need to determine the number of complete cycles by dividing 2011 by 4.

$$\frac{2011}{4} = 502\frac{3}{4}$$

There are 502 complete cycles and $\frac{3}{4}$ of another cycle. $502 \times 4 = 2008$ so 7^{2008} is the last power of 7 in the 502$^{\text{nd}}$ cycle and therefore ends with 01.

To go $\frac{3}{4}$ of the way into the next cycle tells us that the number 7^{2011} ends with the third number in the pattern, namely 43. In fact, we know that 7^{2009} ends with 07, 7^{2010} ends with 49, 7^{2011} ends with 43, and 7^{2012} ends with 01 because they would be the numbers in the 503$^{\text{rd}}$ complete cycle.

Therefore, 7^{2011} ends with the last two digits 43.