Euclidean Geometry
Solutions

1. The area of quadrilateral $ABCD$ is the sum of the areas of $\triangle ABC$ and $\triangle ACD$.

   Since $\triangle ABC$ is right-angled at B, its area equals $\frac{1}{2}(AB)(BC) = \frac{1}{2}(3)(4) = 6$.

   Since $\triangle ABC$ is right-angled at B, by the Pythagorean Theorem,

   $$AC = \sqrt{AB^2 + BC^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

   because $AC > 0$. (We could have also observed that $\triangle ABC$ must be a "3-4-5" triangle.)

   Since $\triangle ACD$ is right-angled at A, by the Pythagorean Theorem,

   $$AD = \sqrt{CD^2 - AC^2} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$$

   because $AD > 0$. (We could have also observed that $\triangle ACD$ must be a "5-12-13" triangle.)

   Thus, the area of $\triangle ACD$ equals $\frac{1}{2}(AC)(AD) = \frac{1}{2}(5)(12) = 30$. Finally, the area of quadrilateral $ABCD$ is thus $6 + 30 = 36$.

2. **Solution 1**

   Suppose that the rectangular prism has dimensions a cm by b cm by c cm.

   Suppose further that one of the faces that is a cm by b cm is the face with area 27 cm^2 and that one of the faces that is a cm by c cm is the face with area 32 cm^2. (Since every pair of non-congruent faces shares exactly one side length, there is no loss of generality in picking these particular variables for these faces.) Therefore, $ab = 27$ and $ac = 32$.

   Further, we are told that the volume of the prism is 144 cm^3, and so $abc = 144$.

   Thus, $bc = \dfrac{a^2b^2c^2}{a^2bc} = \dfrac{(abc)^2}{(ab)(ac)} = \dfrac{144^2}{(27)(32)} = 24$. (We could also note that $abc = 144$ means $a^2b^2c^2 = 144^2$ or $(ab)(ac)(bc) = 144^2$ and so $bc = \dfrac{144^2}{(27)(32)}$.)

   In other words, the third type of face of the prism has area 24 cm^2.

   Thus, since the prism has two faces of each type, the surface area of the prism is equal to $2(27\text{ cm}^2 + 32\text{ cm}^2 + 24\text{ cm}^2)$ or 166 cm^2.

   Solution 2

   Suppose that the rectangular prism has dimensions a cm by b cm by c cm.

   Suppose further that one of the faces that is a cm by b cm is the face with area 27 cm^2 and that one of the faces that is a cm by c cm is the face with area 32 cm^2. (Since every pair of non-congruent faces shares exactly one side length, there is no loss of generality in picking these particular variables for these faces.) Therefore, $ab = 27$ and $ac = 32$.

   Further, we are told that the volume of the prism is 144 cm^3, and so $abc = 144$.

   Since $abc = 144$ and $ab = 27$, we have $c = \dfrac{144}{27} = \dfrac{16}{3}$.

Since $abc = 144$ and $ac = 32$, we have $b = \frac{144}{32} = \frac{9}{2}$.

This means that $bc = \frac{16}{3} \cdot \frac{9}{2} = 24$.

In cm^2, the surface area of the prism equals $2ab + 2ac + 2bc = 2(27) + 2(32) + 2(24) = 166$.

Thus, the surface area of the prism is 166 cm^2.

3. Let the radius of the smaller circle be r cm and let the radius of the larger circle be R cm.

   Thus, the circumference of the smaller circle is $2\pi r$ cm, the circumference of the larger circle is $2\pi R$ cm, the area of the smaller circle is πr^2 cm^2, and the area of the larger circle is πR^2 cm^2.

   Since the sum of the radii of the two circles is 10 cm, we have $r + R = 10$.

   Since the circumference of the larger circle is 3 cm larger than the circumference of the smaller circle, it follows that $2\pi R - 2\pi r = 3$, or $2\pi(R - r) = 3$.

   Then the difference, in cm^2, between the area of the larger circle and the area of the smaller circle is

$$\pi R^2 - \pi r^2 = \pi(R - r)(R + r) = \frac{1}{2}[2\pi(R - r)](R + r) = \frac{1}{2}(3)(10) = 15$$

   Therefore, the difference between the areas is 15 cm^2.

4. Since ABC is a quarter of a circular pizza with centre A and radius 20 cm, we have $AC = AB = 20$ cm. We are also told that $\angle CAB = 90^\circ$ (one-quarter of 360°).

   Since $\angle CAB = 90^\circ$ and A, B and C are all on the circumference of the circle, it follows that CB is a diameter of the pan. (This is a property of circles: if X, Y and Z are three points on a circle with $\angle ZXY = 90^\circ$, then YZ must be a diameter of the circle.)

   Since $\triangle CAB$ is a right-angled isosceles triangle, we have $CB = \sqrt{2}AC = 20\sqrt{2}$ cm. Therefore, the radius of the circular plate is $\frac{1}{2}CB$ or $10\sqrt{2}$ cm. Thus, the area of the circular pan is $\pi(10\sqrt{2}\text{ cm})^2 = 200\pi$ cm^2.

   The area of the slice of pizza is one-quarter of the area of a circle with radius 20 cm, or $\frac{1}{4}\pi(20\text{ cm})^2 = 100\pi$ cm^2.

   Finally, the fraction of the pan that is covered is the area of the slice of pizza divided by the area of the pan, or $\frac{100\pi\text{ cm}^2}{200\pi\text{ cm}^2} = \frac{1}{2}$.

5. **Solution 1**

   Since $\triangle AFD$ is right-angled at F, by the Pythagorean Theorem,

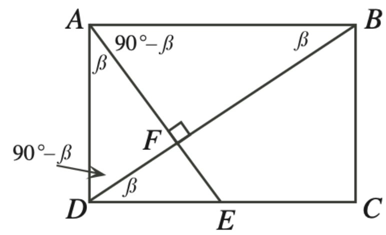
$$AD = \sqrt{AF^2 + FD^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

   since $AD > 0$.

   Let $\angle FAD = \beta$. Since $ABCD$ is a rectangle, we have $\angle BAF = 90^\circ - \beta$.

   Since $\triangle AFD$ is right-angled at F, we have $\angle ADF = 90^\circ - \beta$.

   Since $ABCD$ is a rectangle, we have $\angle BDC = 90^\circ - (90^\circ - \beta) = \beta$.



Therefore, $\triangle BFA$, $\triangle AFD$, and $\triangle DFE$ are all similar as each is right-angled and has either an angle of β or an angle of $90° - \beta$ (and hence both of these angles).

Therefore, $\dfrac{AB}{AF} = \dfrac{DA}{DF}$ and so $AB = \dfrac{4(2\sqrt{5})}{2} = 4\sqrt{5}$.

Also, $\dfrac{FE}{FD} = \dfrac{FD}{FA}$ and so $FE = \dfrac{2(2)}{4} = 1$.

Since $ABCD$ is a rectangle, we have $BC = AD = 2\sqrt{5}$, and $DC = AB = 4\sqrt{5}$.

Finally, the area of quadrilateral $BCEF$ equals the area of $\triangle DCB$ minus the area $\triangle DFE$. Thus, the required area is

$$\frac{1}{2}(DC)(CB) - \frac{1}{2}(DF)(FE) = \frac{1}{2}(4\sqrt{5})(2\sqrt{5}) - \frac{1}{2}(2)(1) = 20 - 1 = 19$$

Solution 2

Since $\triangle AFD$ is right-angled at F, by the Pythagorean Theorem,

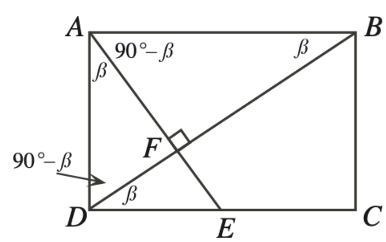
$$AD = \sqrt{AF^2 + FD^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

since $AD > 0$.

Let $\angle FAD = \beta$. Since $ABCD$ is a rectangle, we have $\angle BAF = 90° - \beta$. Since $\triangle BAF$ is right-angled at F, we have $\angle ABF = \beta$.

Since $\triangle AFD$ is right-angled at F, we have $\angle ADF = 90° - \beta$.

Since $ABCD$ is a rectangle, we have $\angle BDC = 90° - (90° - \beta) = \beta$.



Looking at $\triangle AFD$, we see that $\sin\beta = \dfrac{FD}{AD} = \dfrac{2}{2\sqrt{5}} = \dfrac{1}{\sqrt{5}}$, $\cos\beta = \dfrac{AF}{AD} = \dfrac{4}{2\sqrt{5}} = \dfrac{2}{\sqrt{5}}$, and $\tan\beta = \dfrac{FD}{AF} = \dfrac{2}{4} = \dfrac{1}{2}$.

Since $AF = 4$ and $\angle ABF = \beta$, we have $AB = \dfrac{AF}{\sin\beta} = \dfrac{4}{\frac{1}{\sqrt{5}}} = 4\sqrt{5}$.

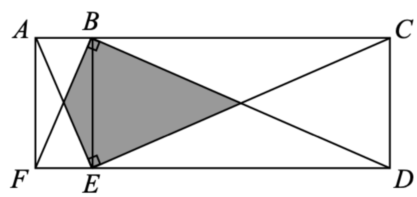
Since $FD = 2$ and $\angle FDE = \beta$, we have $FE = FD \tan\beta = 2 \cdot \frac{1}{2} = 1$.

Since $ABCD$ is a rectangle, we have $BC = AD = 2\sqrt{5}$, and $DC = AB = 4\sqrt{5}$.

Finally, the area of quadrilateral $EFBC$ equals the area of $\triangle DCB$ minus the area $\triangle DFE$. Thus, the required area is

$$\frac{1}{2}(DC)(CB) - \frac{1}{2}(DF)(FE) = \frac{1}{2}(4\sqrt{5})(2\sqrt{5}) - \frac{1}{2}(2)(1) = 20 - 1 = 19$$

6. Join BE.



Since $\triangle FBD$ is congruent to $\triangle AEC$, we have $FB = AE$.

Since $\triangle FAB$ and $\triangle AFE$ are each right-angled, share a common side AF and have equal hypotenuses ($FB = AE$), it follows that these triangles are congruent, and so $AB = FE$.

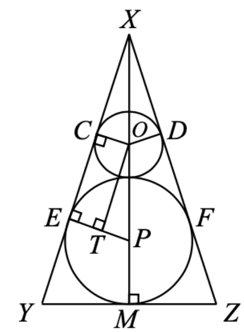
Now $BAFE$ has two right angles at A and F (so AB and FE are parallel) and has equal sides $AB = FE$ so must be a rectangle. This means that $BCDE$ is also a rectangle.

Now the diagonals of a rectangle partition it into four triangles of equal area. (Diagonal AE of the rectangle splits the rectangle into two congruent triangles, which have equal area. The diagonals bisect each other, so the four smaller triangles all have equal area.)

Since one quarter of rectangle $ABEF$ is shaded and one quarter of rectangle $BCDE$ is shaded, it follows that one quarter of the total area is shaded. (If the area of $ABEF$ is x and the area of $BCDE$ is y, then the total shaded area is $\frac{1}{4}x + \frac{1}{4}y$, which is one quarter of the total area $x + y$.)

Since $AC = 200$ and $CD = 50$, the area of rectangle $ACDF$ is $200(50) = 10\,000$, so the total shaded area is $\frac{1}{4}(10\,000) = 2500$.

7. Suppose that M is the midpoint of YZ. Suppose that the centre of the smaller circle is O and the centre of the larger circle is P. Suppose that the smaller circle touches XY at C and XZ at D, and that the larger circle touches XY at E and XZ at F. Join OC, OD and PE.



Since OC and PE are radii that join the centres of circles to points of tangency, it follows that OC and PE are perpendicular to XY.
Construct XM. Since $\triangle XYZ$ is isosceles, XM (which is a median by construction) is an altitude (that is, XM is perpendicular to YZ) and an angle bisector (that is, $\angle MXY = \angle MXZ$).
Now XM passes through O and P. (Since XC and XD are tangents from X to the same circle, we have $XC = XD$. This means that $\triangle XCO$ is congruent to $\triangle XDO$ by side-side-side. This means that $\angle OXC = \angle OXD$ and so O lies on the angle bisector of $\angle CXD$, and so O lies on XM. Using a similar argument, P lies on XM.)
Draw a perpendicular from O to T on PE. Note that OT is parallel to XY (since each is perpendicular to PE) and that $OCET$ is a rectangle (since it has three right angles).

Consider $\triangle XMY$ and $\triangle OTP$. Each triangle is right-angled (at M and at T, respectively). Also, $\angle YXM = \angle POT$. (This is because OT is parallel to XY, since both are perpendicular to PE.) Therefore, $\triangle XMY$ is similar to $\triangle OTP$. Thus, $\dfrac{XY}{YM} = \dfrac{OP}{PT}$.

Now $XY = a$ and $YM = \dfrac{1}{2}b$. Also, OP is the line segment joining the centres of two tangent circles, so $OP = r + R$.

Lastly, $PT = PE - ET = R - r$, since $PE = R$, $ET = OC = r$, and $OCET$ is a rectangle. Therefore,

$$\begin{aligned}
\frac{a}{b/2} &= \frac{R+r}{R-r} \\
\frac{2a}{b} &= \frac{R+r}{R-r} \\
2a(R-r) &= b(R+r) \\
2aR - bR &= 2ar + br \\
R(2a-b) &= r(2a+b) \\
\frac{R}{r} &= \frac{2a+b}{2a-b} \qquad \text{(since } 2a > b \text{ we have } 2a-b \neq 0\text{, and } r > 0\text{)}
\end{aligned}$$

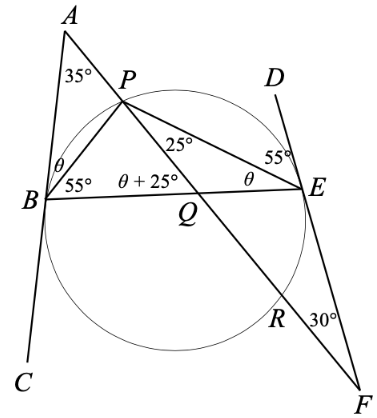
Therefore, $\dfrac{R}{r} = \dfrac{2a+b}{2a-b}$.

8. Let $\angle PEQ = \theta$. Join P to B.

We use the fact that the angle between a tangent to a circle and a chord in that circle that passes through the point of tangency equals the angle inscribed by that chord. We prove this fact below.

More concretely, $\angle DEP = \angle PBE$ (using the chord EP and the tangent through E) and $\angle ABP = \angle PEQ = \theta$ (using the chord BP and the tangent through B).

Now $\angle DEP$ is exterior to $\triangle FEP$ and so $\angle DEP = \angle FPE + \angle EFP = 25^\circ + 30^\circ$, and so $\angle PBE = \angle DEP = 55^\circ$. Furthermore, $\angle AQB$ is an exterior angle of $\triangle PQE$. Thus, $\angle AQB = \angle QPE + \angle PEQ = 25^\circ + \theta$.

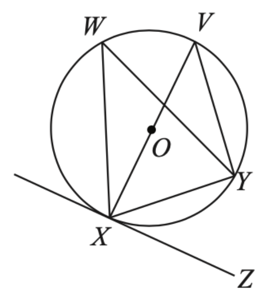


In $\triangle ABQ$, we have $\angle BAQ = 35°$, $\angle ABQ = \theta + 55°$, and $\angle AQB = 25° + \theta$.

Thus, $35° + (\theta + 55°) + (25° + \theta) = 180°$ or $115° + 2\theta = 180°$, and so $2\theta = 65°$. Therefore $\angle PEQ = \theta = \frac{1}{2}(65°) = 32.5°$.

As an addendum, we prove that the angle between a tangent to a circle and a chord in that circle that passes through the point of tangency equals the angle inscribed by that chord.

Consider a circle with centre O and a chord XY, with tangent ZX meeting the circle at X. We prove that if ZX is tangent to the circle, then $\angle ZXY$ equals $\angle XWY$ whenever W is a point on the circle on the opposite side of XY as XZ (that is, the angle subtended by XY on the opposite side of the circle).
We prove this in the case that $\angle ZXY$ is acute. The cases where $\angle ZXY$ is a right angle or an obtuse angle are similar.
Draw diameter XOV and join VY.



Since $\angle ZXY$ is acute, points V and W are on the same arc of chord XY. This means that $\angle XVY = \angle XWY$, since they are angles subtended by the same chord.

Since OX is a radius and XZ is a tangent, it follows that $\angle OXZ = 90°$. Therefore, we have $\angle OXY + \angle ZXY = 90°$.

Since XV is a diameter, we have $\angle XYV = 90°$.

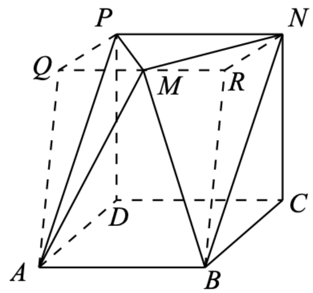
From $\triangle XYV$, we see that $\angle XVY + \angle VXY = 90°$.

But $\angle OXY + \angle ZXY = 90°$ and $\angle XVY + \angle VXY = 90°$ and $\angle OXY = \angle VXY$ tells us that $\angle ZXY = \angle XVY$. This gives us that $\angle ZXY = \angle XWY$, as required.

9. **Solution 1**

Draw a line segment through M in the plane of $\triangle PMN$ parallel to PN and extend this line until it reaches the plane through P, A and D at Q on one side and the plane through N, B and C at R on the other side.

Join Q to P and A. Join R to N and B.



So the volume of solid $ABCDPMN$ equals the volume of solid $ABCDPQRN$ minus the volumes of solids $PMQA$ and $NMRB$.

Solid $ABCDPQRN$ is a trapezoidal prism. This is because NR and BC are parallel (since they lie in parallel planes), which makes $NRBC$ a trapezoid. Similarly, $PQAD$ is a trapezoid. Also, PN, QR, DC, and AB are all perpendicular to the planes of these trapezoids and equal in length, since they equal the side lengths of the squares.

Solids $PMQA$ and $NMRB$ are triangular-based pyramids. We can think of their bases as being $\triangle PMQ$ and $\triangle NMR$. Their heights are each equal to 2, the height of the original solid. (The volume of a triangular-based pyramid equals $\frac{1}{3}$ times the area of its base times its height.)

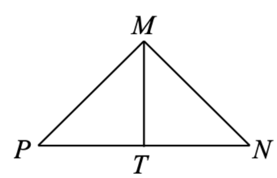
The volume of $ABCDPQRN$ equals the area of trapezoid $NRBC$ times the width of the prism, which is 2.

That is, this volume equals $\frac{1}{2}(NR + BC)(NC)(NP) = \frac{1}{2}(NR + 2)(2)(2) = 2 \cdot NR + 4$.

So we need to find the length of NR.

Consider quadrilateral $PNRQ$. This quadrilateral is a rectangle since PN and QR are perpendicular to the two side planes of the original solid. Thus, NR equals the height of $\triangle PMN$.

Join M to the midpoint T of PN. Since $\triangle PMN$ is isosceles, MT is perpendicular to PN.



Since $NT = \frac{1}{2}PN = 1$ and $\angle PMN = 90^\circ$ and $\angle TNM = 45^\circ$, it follows that $\triangle MTN$ is also right-angled and isosceles with $MT = TN = 1$.

Therefore, $NR = MT = 1$ and so the volume of $ABCDPQRN$ is $2 \cdot 1 + 4 = 6$.

The volumes of solids $PMQA$ and $NMRB$ are equal. Each has height 2 and their bases $\triangle PMQ$ and $\triangle NMR$ are congruent, because each is right-angled (at Q and at R) with $PQ = NR = 1$ and $QM = MR = 1$.

Thus, using the formula above, the volume of each is $\frac{1}{3}\left(\frac{1}{2}(1)(1)\right)2 = \frac{1}{3}$.

Finally, the volume of the original solid equals $6 - 2 \cdot \frac{1}{3} = \frac{16}{3}$.

Solution 2

We determine the volume of $ABCDPMN$ by splitting it into two solids: $ABCDPN$ and $ABNPM$ by slicing along the plane of $ABNP$.

Solid $ABCDPN$ is a triangular prism, since $\triangle BCN$ and $\triangle ADP$ are each right-angled (at C and D), $BC = CN = AD = DP = 2$, and segments PN, DC and AB are perpendicular to each of the triangular faces and equal in length.

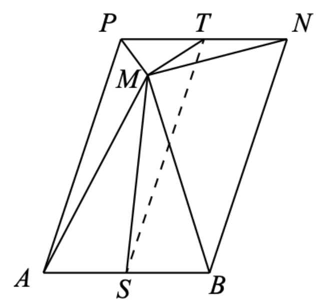
Thus, the volume of $ABCDPN$ equals the area of $\triangle BCN$ times the length of DC. Therefore, $\frac{1}{2}(BC)(CN)(DC) = \frac{1}{2}(2)(2)(2) = 4$. (This solid can also be viewed as "half" of a cube.)

Solid $ABNPM$ is a pyramid with rectangular base $ABNP$. (Note that PN and AB are perpendicular to the planes of both of the side triangular faces of the original solid, that $PN = AB = 2$ and $BN = AP = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, by the Pythagorean Theorem.)

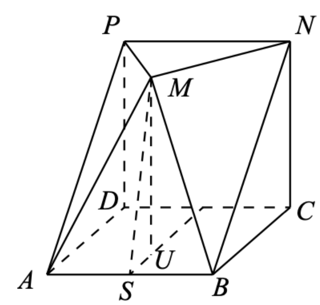
Therefore, the volume of $ABNPM$ equals $\frac{1}{3}(AB)(BN)h = \frac{4\sqrt{2}}{3}h$, where h is the height of the pyramid (that is, the distance that M is above plane $ABNP$).

So we need to calculate h.

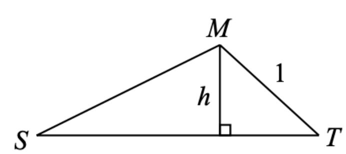
Join M to the midpoint, T, of PN and to the midpoint, S, of AB. Join S and T. By symmetry, M lies directly above ST. Since $ABNP$ is a rectangle and S and T are the midpoints of opposite sides, we have $ST = AP = 2\sqrt{2}$.
Since $\triangle PMN$ is right-angled and isosceles, MT is perpendicular to PN. Since $NT = \frac{1}{2}PN = 1$ and $\angle TNM = 45^\circ$, it follows that $\triangle MTN$ is also right-angled and isosceles with $MT = TN = 1$.



Also, MS is the hypotenuse of the triangle formed by dropping a perpendicular from M to U in the plane of $ABCD$ (a distance of 2) and joining U to S. Since M is 1 unit horizontally from PN, we have $US = 1$.
Thus, $MS = \sqrt{2^2 + 1^2} = \sqrt{5}$ by the Pythagorean Theorem.



We can now consider $\triangle SMT$. h is the height of this triangle, from M to base ST.



Now $h = MT\sin(\angle MTS) = \sin(\angle MTS)$.

By the cosine law in $\triangle SMT$, we have

$$MS^2 = ST^2 + MT^2 - 2(ST)(MT)\cos(\angle MTS)$$

Therefore, $5 = 8 + 1 - 4\sqrt{2}\cos(\angle MTS)$ or $4\sqrt{2}\cos(\angle MTS) = 4$.

Thus, $\cos(\angle MTS) = \dfrac{1}{\sqrt{2}}$ and so $\angle MTS = 45^\circ$ which gives $h = \sin(\angle MTS) = \dfrac{1}{\sqrt{2}}$.

(Alternatively, we note that the plane of $ABCD$ is parallel to the plane of PMN, and so since the angle between plane $ABCD$ and plane $PNBA$ is 45°, it follows that the angle between plane $PNBA$ and plane PMN is also 45°, and so $\angle MTS = 45^\circ$.)

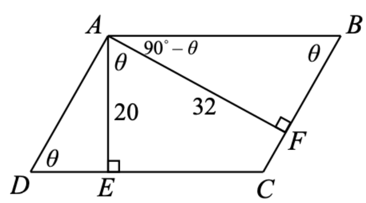
Finally, this means that the volume of $ABNPM$ is $\dfrac{4\sqrt{2}}{3} \cdot \dfrac{1}{\sqrt{2}} = \dfrac{4}{3}$, and so the volume of solid $ABCDPMN$ is $4 + \dfrac{4}{3} = \dfrac{16}{3}$.

10. Let $\angle EAF = \theta$. Since $ABCD$ is a parallelogram, AB and DC are parallel with $AB = DC$, and DA and CB are parallel with $DA = CB$.

Since AE is perpendicular to DC and AB and DC are parallel, it follows that AE is perpendicular to AB. In other words, $\angle EAB = 90^\circ$, and so $\angle FAB = 90^\circ - \theta$.

Since $\triangle AFB$ is right-angled at F and $\angle FAB = 90^\circ - \theta$, we have $\angle ABF = \theta$.

Using similar arguments, we obtain that $\angle DAE = 90^\circ - \theta$ and $\angle ADE = \theta$.



Since $\cos(\angle EAF) = \cos\theta = \dfrac{1}{3}$ and $\cos^2\theta + \sin^2\theta = 1$, we have

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

(Note that $\sin\theta > 0$ since θ is an angle in a triangle.)

In $\triangle AFB$, $\sin\theta = \dfrac{AF}{AB}$ and $\cos\theta = \dfrac{FB}{AB}$.

Since $AF = 32$ and $\sin\theta = \dfrac{2\sqrt{2}}{3}$, we have $AB = \dfrac{AF}{\sin\theta} = \dfrac{32}{2\sqrt{2}/3} = \dfrac{48}{\sqrt{2}} = 24\sqrt{2}$.

Since $AB = 24\sqrt{2}$ and $\cos\theta = \dfrac{1}{3}$, we have $FB = AB\cos\theta = 24\sqrt{2}(\frac{1}{3}) = 8\sqrt{2}$.

In $\triangle AED$, $\sin\theta = \dfrac{AE}{AD}$ and $\cos\theta = \dfrac{DE}{AD}$.

Since $AE = 20$ and $\sin\theta = \dfrac{2\sqrt{2}}{3}$, we have $AD = \dfrac{AE}{\sin\theta} = \dfrac{20}{2\sqrt{2}/3} = \dfrac{30}{\sqrt{2}} = 15\sqrt{2}$.

Since $AD = 15\sqrt{2}$ and $\cos\theta = \dfrac{1}{3}$, we have $DE = AD\cos\theta = 15\sqrt{2}\left(\dfrac{1}{3}\right) = 5\sqrt{2}$. (To calculate AD and DE, we could also have used the fact that $\triangle ADE$ is similar to $\triangle ABF$.)

Finally, the area of quadrilateral $AECF$ equals the area of parallelogram $ABCD$ minus the combined areas of $\triangle AFB$ and $\triangle ADE$.

The area of parallelogram $ABCD$ equals $AB \cdot AE = 24\sqrt{2} \cdot 20 = 480\sqrt{2}$.

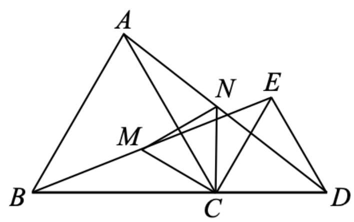
The area of $\triangle AFB$ equals $\frac{1}{2}(AF)(FB) = \frac{1}{2}(32)(8\sqrt{2}) = 128\sqrt{2}$.

The area of $\triangle AED$ equals $\frac{1}{2}(AE)(DE) = \frac{1}{2}(20)(5\sqrt{2}) = 50\sqrt{2}$.

Thus, the area of quadrilateral $AECF$ is $480\sqrt{2} - 128\sqrt{2} - 50\sqrt{2} = 302\sqrt{2}$.

11. **Solution 1**

Consider $\triangle BCE$ and $\triangle ACD$.



Since $\triangle ABC$ is equilateral, we have $BC = AC$. Since $\triangle ECD$ is equilateral, we have that $CE = CD$.

Since BCD is a straight line and $\angle ECD = 60^\circ$, we have $\angle BCE = 180^\circ - \angle ECD = 120^\circ$.

Since BCD is a straight line and $\angle BCA = 60^\circ$, we have $\angle ACD = 180^\circ - \angle BCA = 120^\circ$.

Therefore, $\triangle BCE$ is congruent to $\triangle ACD$ ("side-angle-side").

Since $\triangle BCE$ and $\triangle ACD$ are congruent and CM and CN are line segments drawn from the corresponding vertex (C in both triangles) to the midpoint of the opposite side, we have $CM = CN$.

Since $\angle ECD = 60^\circ$, $\triangle ACD$ can be obtained by rotating $\triangle BCE$ through an angle of 60° clockwise about C. This means that after this 60° rotation, CM coincides with CN. In other words, $\angle MCN = 60^\circ$.

But since $CM = CN$ and $\angle MCN = 60^\circ$, we have

$$\angle CMN = \angle CNM = \frac{1}{2}(180^\circ - \angle MCN) = 60^\circ$$

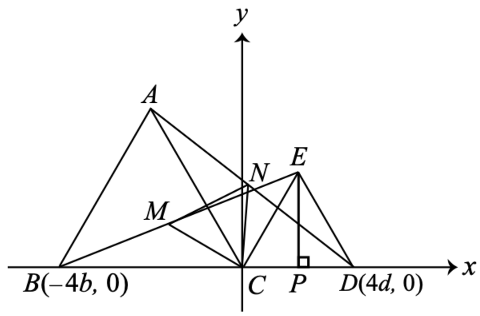
Therefore, $\triangle MNC$ is equilateral, as required.

Solution 2

We prove that $\triangle MNC$ is equilateral by introducing a coordinate system.

Suppose that C is at the origin $(0, 0)$ with BCD along the x-axis, with B having coordinates $(-4b, 0)$ and D having coordinates $(4d, 0)$ for some real numbers $b, d > 0$.

Drop a perpendicular from E to P on CD.



Since $\triangle ECD$ is equilateral, P is the midpoint of CD.

Since C has coordinates $(0,0)$ and D has coordinates $(4d,0)$, it follows that the coordinates of P are $(2d,0)$.

Since $\triangle ECD$ is equilateral, we have $\angle ECD = 60^\circ$ and so $\triangle EPC$ is a 30°-60°-90° triangle and so $EP = \sqrt{3}CP = 2\sqrt{3}d$.

Therefore, the coordinates of E are $(2d, 2\sqrt{3}d)$.

In a similar way, we can show that the coordinates of A are $(-2b, 2\sqrt{3}b)$.

Now M is the midpoint of $B(-4b,0)$ and $E(2d,2\sqrt{3}d)$, and therefore, the coordinates of M are $\left(\frac{1}{2}(-4b+2d), \frac{1}{2}(0+2\sqrt{3}d)\right)$ or $(-2b+d, \sqrt{3}d)$.

Also, N is the midpoint of $A(-2b, 2\sqrt{3}b)$ and $D(4d,0)$, and therefore, the coordinates of N are $\left(\frac{1}{2}(-2b+4d), \frac{1}{2}(2\sqrt{3}b+0)\right)$ or $(-b+2d, \sqrt{3}b)$.

To show that $\triangle MNC$ is equilateral, we show that $CM = CN = MN$ or equivalently that $CM^2 = CN^2 = MN^2$:

$$\begin{aligned}
CM^2 &= (-2b+d-0)^2 + (\sqrt{3}d-0)^2 \\
&= (-2b+d)^2 + (\sqrt{3}d)^2 \\
&= 4b^2 - 4bd + d^2 + 3d^2 \\
&= 4b^2 - 4bd + 4d^2
\end{aligned}$$

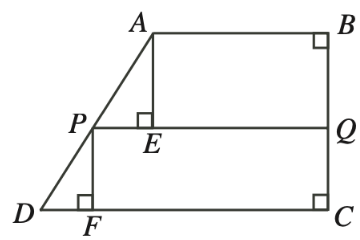
$$\begin{aligned}
CN^2 &= (-b+2d-0)^2 + (\sqrt{3}b-0)^2 \\
&= (-b+2d)^2 + (\sqrt{3}b)^2 \\
&= b^2 - 4bd + 4d^2 + 3b^2 \\
&= 4b^2 - 4bd + 4d^2
\end{aligned}$$

$$\begin{aligned}
MN^2 &= ((-2b+d)-(-b+2d))^2 + (\sqrt{3}d - \sqrt{3}b)^2 \\
&= (-b-d)^2 + 3(d-b)^2 \\
&= b^2 + 2bd + d^2 + 3d^2 - 6bd + 3b^2 \\
&= 4b^2 - 4bd + 4d^2
\end{aligned}$$

Therefore, $CM^2 = CN^2 = MN^2$ and so $\triangle MNC$ is equilateral, as required.

12. Since PQ is parallel to AB, it is parallel to DC and is perpendicular to BC.

    Drop perpendiculars from A to E on PQ and from P to F on DC.



Then $ABQE$ and $PQCF$ are rectangles. Thus, $EQ = x$, which means that $PE = r - x$ and $FC = r$, which means that $DF = y - r$.

Let $BQ = b$ and $QC = c$. Thus, $AE = b$ and $PF = c$.

The area of trapezoid $ABQP$ is $\frac{1}{2}(x + r)b$.

The area of trapezoid $PQCD$ is $\frac{1}{2}(r + y)c$.

Since these areas are equal, we have $\frac{1}{2}(x + r)b = \frac{1}{2}(r + y)c$, which gives $\frac{x + r}{r + y} = \frac{c}{b}$.

Since AE is parallel to PF, we have $\angle PAE = \angle DPF$ and $\triangle AEP$ is similar to $\triangle PFD$.

Thus, $\frac{AE}{PE} = \frac{PF}{DF}$ which gives $\frac{b}{r - x} = \frac{c}{y - r}$ or $\frac{c}{b} = \frac{y - r}{r - x}$.

Combining $\frac{x + r}{r + y} = \frac{c}{b}$ and $\frac{c}{b} = \frac{y - r}{r - x}$ gives $\frac{x + r}{r + y} = \frac{y - r}{r - x}$ or $(x + r)(r - x) = (r + y)(y - r)$.

From this, we get $r^2 - x^2 = y^2 - r^2$ or $2r^2 = x^2 + y^2$, as required.

13. Suppose that the parallel line segments EF and WX are a distance of x apart.

    This means that the height of trapezoid $EFXW$ is x.

    Since the side length of square $EFGH$ is 10 and the side length of square $WXYZ$ is 6, it follows that the distance between parallel line segments ZY and HG is $10 - 6 - x$ or $4 - x$.

    Recall that the area of a trapezoid equals one-half times its height times the sum of the lengths of the parallel sides.

    Thus, the area of trapezoid $EFXW$ is $\frac{1}{2}x(EF + WX) = \frac{1}{2}x(10 + 6) = 8x$.

    Also, the area of trapezoid $GHZY$ is $\frac{1}{2}(4 - x)(HG + ZY) = \frac{1}{2}(4 - x)(10 + 6) = 32 - 8x$.

    Therefore, the sum of the areas of trapezoids $EFXW$ and $GHZY$ is $8x + (32 - 8x) = 32$.

    This sum is a constant and does not depend on the position of the inner square within the outer square, as required.