Properties of Numbers
Solutions

1. Suppose that a palindrome p is the sum of the three consecutive integers $a-1$, a, $a+1$.

   In this case, $p=(a-1)+a+(a+1)=3a$, so p is a multiple of 3.

   The largest palindromes less than 200 are 191, 181, 171.

   Note that 191 and 181 are not divisible by 3, but 171 is divisible by 3. We can easily check this using the divisibility by 3 test. For each of these integers, the sum of their digits is 11, 10 and 9, respectively. Only 9 is divisible by 3 and so 171 is the only one that is divisible by 3.

   The integer 171 can be written as $56+57+58$, so 171 is the largest palindrome less than 200 that is the sum of three consecutive integers.

2. Suppose that n has digits AB. Then $n=10A+B$. The average of the digits of n is $\dfrac{A+B}{2}$. Putting a decimal point between the digits of n is equivalent to dividing n by 10, so the resulting number is $\dfrac{10A+B}{10}$.

   So we want to determine A and B so that

$$\begin{aligned}\frac{10A+B}{10}&=\frac{A+B}{2}\\ 10A+B&=5(A+B)\\ 5A&=4B\end{aligned}$$

   Since A and B are digits such that $5A=4B$, we have $A=4$ and $B=5$ is the only possibility. Therefore, $n=45$. (We can quickly check that the average of the digits of n is 4.5, the number obtained by putting a decimal point between the digits of n.)

3. **Solution 1**

   The integer equal to 10^{20} consists of the digit 1 followed by 20 0s. The integer equal to $10^{20}-1$ thus consists of 20 9s.

   Now, $n=10^{20}-20$ is 19 less than $10^{20}-1$ which is the integer that consists of 20 9s.

   So $n=10^{20}-20=99\cdots980$, where this integer has 18 9s.

   Therefore, the sum of the digits of n is $18(9)+8+0=162+8=170$.

   Solution 2

   Since $10^{20}-20=10(10^{19}-2)$ and $10^{19}-2=99\cdots98$ (where this integer has 18 9s), we have $10^{20}-20=99\cdots980$, where this integer has 18 9s.

   Therefore, the sum of the digits of n is $18(9)+8+0=162+8=170$.

4. The *parity* of an integer is whether it is even or odd.

   Since the Fibonacci sequence begins $1,1,2,3,5,8,13,21,\ldots$, it follows that the parities of the first eight terms are Odd, Odd, Even, Odd, Odd, Even, Odd, Odd.

   In the sequence, if x and y are consecutive terms, then the next term is $x+y$.

In general, suppose that x and y are integers. If x is even and y is even, then $x+y$ is even. If x is even and y is odd, then $x+y$ is odd. If x is odd and y is even, then $x+y$ is odd. If x is odd and y is odd, then $x+y$ is even. Therefore, the parities of two consecutive terms x and y in the Fibonacci sequence determine the parity of the following term $x+y$.

Also, once there are two consecutive terms whose parities match the parities of two earlier consecutive terms in the sequence, then the parities will repeat in a cycle. In particular, the parities of the fourth and fifth terms (Odd, Odd) are the same as the parities of the first and second terms (Odd, Odd). Therefore, the parities in the sequence repeat the cycle Odd, Odd, Even. This cycle has length 3. Therefore, the 99th term in the Fibonacci sequence ends one of these cycles, since 99 is a multiple of 3. In particular, the 99th term ends the 33rd cycle.

Each cycle contains two odd terms. Therefore, the first 99 terms in the sequence include $2 \times 33 = 66$ odd terms.

Finally, the 100th term in the sequence begins a new cycle, so it is odd. Therefore, the first 100 terms include $66 + 1 = 67$ odd terms.

5. Since $900 = 30^2$ and $30 = 2 \times 3 \times 5$, we have $900 = 2^2 3^2 5^2$.

   The positive divisors of 900 are those integers of the form $d = 2^a 3^b 5^c$, where each of a, b, c is 0, 1 or 2.

   For d to be a perfect square, the exponent on each prime factor in the prime factorization of d must be even. Thus, for d to be a perfect square, each of a, b, c must be 0 or 2.

   There are two possibilities for each of a, b, c so $2 \times 2 \times 2 = 8$ possibilities for d.

   These are $2^0 3^0 5^0 = 1$, $2^2 3^0 5^0 = 4$, $2^0 3^2 5^0 = 9$, $2^0 3^0 5^2 = 25$, $2^2 3^2 5^0 = 36$, $2^2 3^0 5^2 = 100$, $2^0 3^2 5^2 = 225$, and $2^2 3^2 5^2 = 900$.

   Thus, 8 of the positive divisors of 900 are perfect squares.

6. Since each list contains 6 consecutive positive integers and the smallest integers in the lists are a and b, it follows that the positive integers in the first list are $a, a+1, a+2, a+3, a+4, a+5$ and the positive integers in the second list are $b, b+1, b+2, b+3, b+4, b+5$. Note that $1 \leq a < b$.

   We first determine the pairs (a, b) for which 49 will appear in the third list, then determine which of these pairs give a third list that contains no multiple of 64, and then finally keep only those pairs for which there is a number in the third list larger than 75.

   The first bullet tells us that 49 is the product of an integer in the first list and an integer in the second list.

   Since $49 = 7^2$ and 7 is prime, it follows that these integers are either 1 and 49 or 7 and 7.

   If 1 is in one of the lists, then either $a = 1$ or $b = 1$. Since $1 \leq a < b$, it must be that $a = 1$.

   If 49 is in the second list, then one of $b, b+1, b+2, b+3, b+4, b+5$ equals 49, and so $44 \leq b \leq 49$.

   Therefore, for 1 and 49 to appear in the two lists, then (a, b) must be one of

   $$(1, 49), (1, 48), (1, 47), (1, 46), (1, 45), (1, 44)\ .$$

   If 7 appears in the first list, then one of $a, a+1, a+2, a+3, a+4, a+5$ equals 7, so $2 \leq a \leq 7$. Similarly, if 7 appears in the second list, then $2 \leq b \leq 7$.

Therefore, for 7 to appear in both lists, then, knowing that $a < b$, then (a, b) must be one of

$$(2,3),(2,4),(2,5),(2,6),(2,7),(3,4),(3,5),(3,6),(3,7),(4,5),(4,6),(4,7),(5,6),(5,7),(6,7)\ .$$

The second bullet tells us that no pair of numbers in the first and second lists have a product that is a multiple of 64.

Given that the possible values of a and b are $1, 2, 3, 4, 5, 6, 7, 44, 45, 46, 47, 48, 49$, then the possible integers in the two lists are those integers from 1 to 12, inclusive, and from 44 to 54, inclusive. (For example, if the first number in one list is 7, then the remaining numbers in this list are 8, 9, 10, 11, 12.)

There is no multiple of 32 or 64 in these lists.

Thus, for a pair of integers from these lists to have a product that is a multiple of 64, one is a multiple of 4 and the other is a multiple of 16, or both are multiples of 8.

If $(a, b) = (1, 48), (1, 47), (1, 46), (1, 45), (1, 44)$, then 4 appears in the first list and 48 appears in the second list; these have a product of 192, which is $3 \cdot 64$.

If $(a, b) = (1, 49)$, there is a multiple of 4 but not of 8 in the first list, and a multiple of 4 but not of 8 in the second list, so there is no multiple of 64 in the third list.

If $(a, b) = (3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7)$, then 8 appears in both lists, so 64 appears in the third list.

If $(a, b) = (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$, then there is no multiple of 8 or 16 in the first list and no multiple of 16 in the second list, so there is no multiple of 64 in the third list.

Therefore, after considering the first two bullets, the possible pairs (a, b) are $(1, 49), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$.

The third bullet tells us that there is at least one number in the third list that is larger than 75.

Given the possible pairs (a, b) are $(1, 49), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$, the corresponding pairs of largest integers in the lists are $(6, 54), (7, 8), (7, 9), (7, 10), (7, 11), (7, 12)$.

The corresponding largest integers in the third list are the products of the largest integers in the two lists; these products are $324, 56, 63, 70, 77, 84$, respectively.

Therefore, the remaining pairs (a, b) are $(1, 49), (2, 6), (2, 7)$.

Having considered the three conditions, the possible pairs (a, b) are $(1, 49), (2, 6), (2, 7)$.

7. First, we determine the perfect squares between 1300 and 1400 and between 1400 and 1500. Since $\sqrt{1300} \approx 36.06$, the first perfect square larger than 1300 is $37^2 = 1369$. The next perfect squares are $38^2 = 1444$ and $39^2 = 1521$.

   Since Charles was born between 1300 and 1400 in a year that was a perfect square, Charles must have been born in 1369.

   Since Louis was born between 1400 and 1500 in a year that was a perfect square, Louis must have been born in 1444.

   Suppose that on April 7 in some year, Charles was m^2 years old and Louis was n^2 years old for some positive integers m and n. Thus, Charles was m^2 years old in the year $1369 + m^2$ and Louis was n^2 years old in the year $1444 + n^2$.

Since these expressions represent the same years, we have that $1369 + m^2 = 1444 + n^2$, or $m^2 - n^2 = 1444 - 1369 = 75$. In other words, we want to find two perfect squares less than 110 (since their ages are less than 110) whose difference is 75.

The perfect squares less than 110 are $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$. The two that differ by 75 are 100 and 25. Thus, $m^2 = 100$ and $n^2 = 25$.

This means that the year in which the age of each of Charles and Louis was a perfect square was the year $1369 + 100 = 1469$.

8. Since $10^y \neq 0$, the equation $\dfrac{1}{32} = \dfrac{x}{10^y}$ is equivalent to $10^y = 32x$.

   So the given question is equivalent to asking for the smallest positive integer x for which $32x$ equals a positive integer power of 10.

   Now $32 = 2^5$ and so $32x = 2^5x$. For $32x$ to equal a power of 10, each factor of 2 must be matched with a factor of 5. Therefore, x must be divisible by 5^5 (that is, x must include at least 5 powers of 5), and so $x \geq 5^5 = 3125$.

   But $32(5^5) = 2^5 5^5 = 10^5$, and so if $x = 5^5 = 3125$, then $32x$ is indeed a power of 10, namely 10^5. This tells us that the smallest positive integer x for which $\dfrac{1}{32} = \dfrac{x}{10^y}$ for some positive integer y is $x = 5^5 = 3125$.

9. Since the average of three consecutive multiples of 3 is a, a is the middle of these three integers, and so the integers are $a - 3, a, a + 3$.

   Since the average of four consecutive multiples of 4 is $a + 27$, we have that $a + 27$ is halfway in between the second and third of these multiples (which differ by 4), so the second and third of the multiples are $(a + 27) - 2 = a + 25$ and $(a + 27) + 2 = a + 29$. Therefore, the four integers are $a + 21, a + 25, a + 29, a + 33$.

   The smallest of these seven integers is $a - 3$ and the largest is $a + 33$.

   The average of these two integers is $\frac{1}{2}(a - 3 + a + 33) = \frac{1}{2}(2a + 30) = a + 15$.

   Since $a + 15 = 42$, we have $a = 27$.

10. First, we factor the left side of the given equation to obtain $a(a^2 + 2b) = 2013$.

    Next, we factor the integer 2013 as $2013 = 3 \times 671 = 3 \times 11 \times 61$. Note that each of 3, 11 and 61 is prime, so we can factor 2013 no further. (We can find the factors of 3 and 11 using tests for divisibility by 3 and 11, or by systematic trial and error.)

    Since $2013 = 3 \times 11 \times 61$, the positive divisors of 2013 are

    $$1, 3, 11, 33, 61, 183, 671, 2013$$

    Since a and b are positive integers, a and $a^2 + 2b$ are both positive integers.

    Since a and b are positive integers, we have $a^2 \geq a$ and $2b > 0$, so $a^2 + 2b > a$.

    Since $a(a^2 + 2b) = 2013$, a and $a^2 + 2b$ must be a divisor pair of 2013 (that is, a pair of positive integers whose product is 2013) with $a < a^2 + 2b$.

We make a table of the possibilities:

a	a^2+2b	$2b$	b
1	2013	2012	1006
3	671	662	331
11	183	62	31
33	61	-1028	N/A

Note that the last case is not possible, since b must be positive. Therefore, the three pairs of positive integers that satisfy the equation are $(1,1006)$, $(3,331)$, $(11,31)$. (We can verify by substitution that each is a solution of the original equation.)

11. Suppose that the auditorium with these properties has r rows and c columns of chairs. Then there are rc chairs in total.

Each chair is empty, is occupied by a teacher, or is occupied by a student.

Since there are 14 teachers in each row, there are $14r$ chairs occupied by teachers. Since there are 10 students in each column, there are $10c$ chairs occupied by students. Since there are exactly 3 empty chairs, the total number of chairs can also be written as $14r + 10c + 3$. Therefore, $rc = 14r + 10c + 3$.

We proceed to find all pairs of positive integers r and c that satisfy this equation. We note that since there are 14 teachers in each row, there must be at least 14 columns (that is, $c \geq 14$) and since there are 10 students in each column, there must be at least 10 rows (that is, $r \geq 10$).

Manipulating the equation,

$$\begin{aligned} rc &= 14r + 10c + 3 \\ rc - 14r &= 10c + 3 \\ r(c-14) &= 10c + 3 \\ r &= \frac{10c+3}{c-14} \\ r &= \frac{10c - 140 + 143}{c-14} \\ r &= \frac{10c-140}{c-14} + \frac{143}{c-14} \\ r &= 10 + \frac{143}{c-14} \end{aligned}$$

Since r is an integer, $10 + \dfrac{143}{c-14}$ is an integer, and so $\dfrac{143}{c-14}$ must be an integer.

Therefore, $c - 14$ is a divisor of 143. Since $c \geq 14$, we have $c - 14 \geq 0$, and so $c - 14$ is a positive divisor of 143.

Since $143 = 11 \times 13$, its positive divisors are $1, 11, 13, 143$.

We make a table of the possible values of $c - 14$ along with the resulting values of c, r (calculated using $r = 10 + \dfrac{143}{c-14}\Big)$ and rc:

$c-14$	c	r	rc
1	15	153	2295
11	25	23	575
13	27	21	567
143	157	11	1727

Therefore, the four possible values for rc are $567, 575, 1727, 2295$. That is, the smallest possible number of chairs in the auditorium is 567. (Can you create a grid with 27 columns and 21 rows that has the required properties?)