Canadian
Mathematics
Competition

An activity of The Centre for Education
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2003 Solutions
Pascal Contest (Grade 9)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

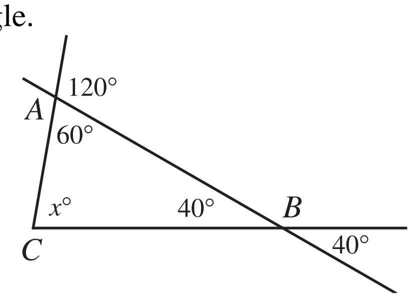
2003 Pascal Contest Solutions

1. Calculating, $\sqrt{169} - \sqrt{25} = 13 - 5 = 8$. ANSWER: (A)

2. Looking at the first few terms in the sequence, we can see that to get from one term to the next, we multiply by 3 (since $6 = 3(2)$, $18 = 3(6)$, etc.) So the missing term is $3(54) = 162$. (To check that this is correct, we can see that $486 = 3(162)$.)
[The term "geometric sequence" means that every term after the first is obtained from the previous term by multiplying by the same number. In this question, the pattern can be guessed and filled in without knowing this definition.]
ANSWER: (D)

3. Using the BEDMAS order of operations,
$\dfrac{6+6\times 3-3}{3} = \dfrac{6+18-3}{3} = \dfrac{21}{3} = 7$. ANSWER: (B)

4. We label the three points of intersection as A, B and C, as shown.
Then $\angle ABC = 40^\circ$ since it is equal to its opposite angle.
Also, $\angle CAB = 60^\circ$ since it is the supplement of a 120° angle.
In ΔABC,
$$x^\circ + 60^\circ + 40^\circ = 180^\circ$$
$$x = 80$$



ANSWER: (E)

5. *Solution 1*
Using exponent laws,
$$\frac{2^8}{8^2} = \frac{2^8}{\left(2^3\right)^2} = \frac{2^8}{2^6} = 2^2 = 4.$$

Solution 2
Calculating directly,
$$\frac{2^8}{8^2} = \frac{256}{64} = 4.$$

ANSWER: (C)

6. Evaluating each of the 5 choices,

(A) $\frac{6^2}{10} = \frac{36}{10} = \frac{18}{5}$ (B) $\frac{1}{5}[6(3)] = \frac{1}{5}[18] = \frac{18}{5}$ (C) $\frac{18+1}{5+1} = \frac{19}{6} \neq \frac{18}{5}$

(D) $3.6 = \frac{36}{10} = \frac{18}{5}$ (E) $\sqrt{\frac{324}{25}} = \sqrt{\frac{18^2}{5^2}} = \frac{18}{5}$

Therefore, the only choice not equal to $\frac{18}{5}$ is (C). ANSWER: (C)

7. Starting from the bottom of the diagram, we first determine the value of F. From the conditions given, either $F - 7 = 3$ or $7 - F = 3$. Since F must be one of 1, 2, 4, 5, 6, and 8, then $F = 4$, giving

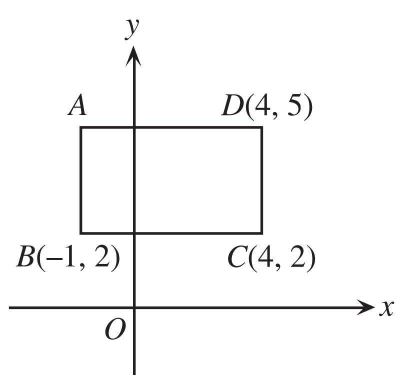
$$\begin{array}{ccccccc} A & & 10 & & B & & C \\ & D & & 9 & & E & \\ & & 7 & & 4 & & \\ & & & 3 & & & \end{array}$$

Similarly, we can determine that $E = 5$ and $D = 2$, giving

$$\begin{array}{ccccccc} A & & 10 & & B & & C \\ & 2 & & 9 & & 5 & \\ & & 7 & & 4 & & \\ & & & 3 & & & \end{array}$$

and then $A = 8$, $B = 1$, and $C = 6$. (We notice that this does use each of the six possibilities exactly once.)
Thus, $A + C = 14$. ANSWER: (E)

8. Since the sides of the rectangle are parallel to the axes, we can determine the side lengths by taking the difference of the appropriate coordinates.
The length of BC is the difference of the x-coordinates, that is $4 - (-1) = 5$.
The length of DC is the difference of the y-coordinates, that is $5 - 2 = 3$.
Therefore, the area of the rectangle is $3 \times 5 = 15$.



ANSWER: (A)

9. Every prime number, with the exception of 2, is an odd number. To write an odd number as the sum of two whole numbers, one must be an even number and one must be an odd number.
So in this case, we want to write a prime as the sum of an odd prime number and an even prime number. Since the only even prime number is 2, then we want to write a prime number as 2 plus another prime number.
From highest to lowest, the prime numbers less than 30 are 29, 23, 19, 17, 13, 11, 7, 5, 3 and 2.

The largest prime less than 30 which is 2 more than another prime number is 19.

ANSWER: (C)

10. We write out each of the five choices to 8 decimal places:
(A) $3.2571 = 3.25710000...$
(B) $3.\overline{2571} = 3.25712571...$
(C) $3.2\overline{571} = 3.25715715...$
(D) $3.25\overline{71} = 3.25717171...$
(E) $3.257\overline{1} = 3.25711111...$
These five real numbers agree to four decimals, but are all different in the fifth decimal place. Therefore, $3.25\overline{71}$ is the largest.

ANSWER: (D)

11. Substituting $x = 2$ and $y = -3$, we obtain

$$\begin{aligned} 2(2)^2 + k(2)(-3) &= 4 \\ 8 - 6k &= 4 \\ 4 &= 6k \\ k &= \tfrac{2}{3} \end{aligned}$$

ANSWER: (A)

12. From the first exchange rate, 1 calculator is worth 100 rulers.
From the second exchange rate, 100 rulers are worth $\frac{100}{10}(30) = 300$ compasses.
From the third exchange rate, 300 compasses are worth $\frac{300}{25}(50) = 600$ protractors.
From this, we can see that 600 protractors are equivalent to 1 calculator.

ANSWER: (B)

13. If we look at the top left four squares, we see that they must all be coloured a different colour, since they all have a vertex in common, so no two can be the same colour.
So we need at least 4 colours. Can we colour the 15 squares with only 4 colours?
If we try to do this, we can see that this is possible by taking a block of four squares, colouring the squares different colours, and shifting this block around the grid. (In the diagram, different numbers represent the four different colours.)
What would happen if the grid were bigger than 3 by 5? Would 4 colours still be enough? This problem is related to a famous math problem called the "Four Colour Problem". Try looking this up on the Web!

1	2	1	2	1
3	4	3	4	3
1	2	1	2	1

ANSWER: (B)

14. *Solution 1*

Since x and y have to be positive integers and add up to 5, it is easy to make a chart of the possibilities:

x	y	$2x-y$
1	4	–2
2	3	1
3	2	4
4	1	7

Of the choices, only –2 is a possibility.

Solution 2

We can rewrite $2x-y=3x-x-y=3x-(x+y)=3x-5$.

Of the five possibilities, the only one that is 5 less than a multiple of 3 is –2.

ANSWER: (D)

15. *Solution 1*

We can find the area of square *KLMN* by subtracting the areas of the four triangles *KAN*, *NDM*, *MCL* and *LBK* from the area of square *ABCD*.

To do this, we note that square *ABCD* has a side length of 6, and that each of the four triangles is right-angled and has one leg of length 2 and the other of length 4. (A "leg" of right-angled triangle is a side that is not the hypotenuse.)

So the area of square *KLMN* is $6^2-4\left[\frac{1}{2}(2)(4)\right]=36-4[4]=20$ square units.

Solution 2

Since *KLMN* is a square, its area is the square of its side length; in particular, the area is equal to NM^2.

To calculate its side length, we look at right-angled triangle *DNM* and calculate NM^2 using Pythagoras:

$$NM^2=ND^2+DM^2$$

$$NM^2=2^2+4^2$$

$$NM^2=20$$

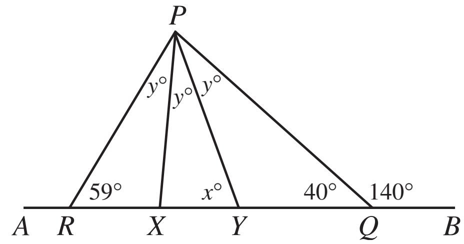
Therefore, the area of the square is 20 square units.

ANSWER: (D)

16. Since n can be any integer, let us choose $n=0$. Then the values of the 5 integers are 3, -9, -4, 6, and -1. When we arrange these from smallest to largest, we get -9, -4, -1, 3, and 6, so the number in the middle is -1, which is $n-1$.

ANSWER: (E)

17. We add the extra labels to the diagram, as shown.



Then $\angle PQR = 180^\circ - \angle PQB = 40^\circ$.
Considering ΔPQR,

$$59^\circ + 3y^\circ + 40^\circ = 180^\circ$$
$$3y = 81$$
$$y = 27$$

We next look at ΔPRY, where

$$59^\circ + 2y^\circ + x^\circ = 180^\circ$$
$$x = 180 - 59 - 2(27)$$
$$x = 67$$

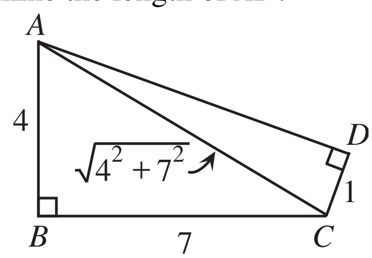
ANSWER: (A)

18. The average of a list of numbers is their sum divided by how many numbers are in the list. Thus if n numbers have an average of 7 their sum is $7n$.
When -11 is added to the list of numbers, there are then $n+1$ numbers whose sum is $7n-11$. Using this, we obtain

$$\frac{7n-11}{n+1} = 6$$
$$7n - 11 = 6n + 6$$
$$n = 17$$

ANSWER: (E)

19. If we join A to C, then the quadrilateral is divided into two right-angled triangles.
The area of triangle ABC is $\frac{1}{2}(4)(7) = 14$.
To find the area of triangle ADC, first we need to determine the length of AD.
Using Pythagoras in both triangles,



$$AD^2 = AC^2 - DC^2$$
$$AD^2 = \left(AB^2 + BC^2\right) - 1^2$$
$$AD^2 = 7^2 + 4^2 - 1^2$$
$$AD^2 = 64$$
$$AD = 8$$

The area of triangle ADC is thus $\frac{1}{2}(1)(8) = 4$.
Therefore, the total area of quadrilateral $ABCD$ is the sum of the areas of the two triangles, or $14 + 4 = 18$.

ANSWER: (C)

20. *Solution 1*
Suppose instead of using the digits 0, 2, 4, 6, 8, that Evenlanders use the digits 0, 1, 2, 3, 4 (each of the previous digits divided by 2).
Then we can see that the Evenlanders' numbers system is our base 5 number system, with all of the digits doubled.

Writing 111 as a sum of powers of 5, we see that $111 = 4\left(5^2\right) + 2\left(5^1\right) + 1\left(5^0\right)$, or 111 can be written as 421 in base 5. Doubling the digits, the Evenlanders write 842 for the integer 111.

Solution 2

In order to determine the Evenlanders' version of 111, we need to find a pattern in the Evenlanders' numbers. So we write out the first several:

2, 4, 6, 8, 20, 22, 24, 26, 28, 40, 42, 44, 46, 48, 60, 62, 64, 66, 68, 80, …

which correspond to the integers 1 through 20. We can see from these that the last digit of the Evenlanders' numbers has a 5 digit cycle (2, 4, 6, 8, 0). Since we are looking for their version of 111, the last digit must be 2.

In order to figure out the other digits, we need to extend our pattern. So continuing to count, we get

80, 82, 84, 86, 88, 200, 202, 204, 206, 208, 220, …

for the integers 20 through 30.

The numbers will then continue to have three digits with first digit 2 as the last two digits go from 00 to 88. After 288 will come 400 and the cycle begins again.

How many three digit numbers beginning with 2 do the Evenlanders have? Since these are the numbers 200 through 288, it is the same number as counting 00 through 88, which is 25 numbers, since 88 corresponds to the integer 24.

So 200 represents the integer 25, which means that 400 represents the integer 50, 600 will represent the integer 75, and 800 will represent the integer 100.

To get from the integer 100 to the integer 111, we can either count as Evenlanders from 800 upwards, or we can look for the 11th number in our original list, which is 42 (which represents the integer 11), and put an 8 at the front.

Therefore, the Evenlanders' version of 111 is 842.

ANSWER: (D)

21. Since each light turns red 10 seconds after the preceding one, and the lights are all on cycles of equal time, then each light will turn green 10 seconds after the preceding one. So let's say that the first light turns green at time 0 seconds. The question to ask is, "When will the last light turn green?"

Since it is the eighth light, it will turn green at time 70 seconds (seven intervals of 10 seconds later). At this point, the first light is still green, since it remains green for 1.5 minutes or 90 seconds, and will remain green for another 20 seconds.

(This first light will then turn yellow at time 90 seconds, and then red at time 93 seconds, and then each of the other lights will turn red until the eighth light turns red at time 163 seconds, when the first light is still red.)

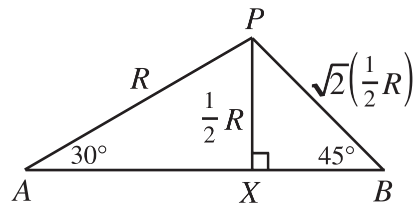
So the only time when all eight lights are green is the interval above, and so the longest interval of time when all the lights are green is 20 seconds.

(Note that the length of time that the light was yellow has no relevance to the situation.)

ANSWER: (C)

22. *Solution 1*

Join A to B (meeting PQ at X) and consider the triangle APB.



By symmetry, $\angle PAB = 30^\circ$, $\angle PBA = 45^\circ$ and PX is perpendicular to AB.

Since $AP = R$ and ΔAPX is a 30-60-90 triangle, then $PX = \frac{1}{2}R$.

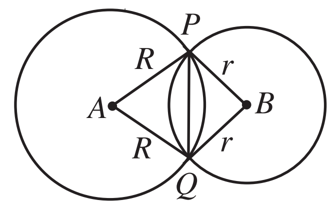
Since ΔBPX is a 45-45-90 triangle, then $PB = \sqrt{2}\left(\frac{1}{2}R\right)$.

Therefore, $r = \sqrt{2}\left(\frac{1}{2}R\right)$ or $r^2 = \frac{1}{2}R^2$.

Since the two circles have areas of πr^2 and πR^2, then the ratio of their areas is 2 : 1.

Solution 2

Suppose that the radius of the circle with centre A is R, and the radius of the circle with centre B is r. Join P to Q.



Since PQ is a chord in each of the two circles, we will try to find the length of PQ in terms of R and in terms of r, in order to find a relationship between R and r.

Consider ΔAPQ. Then $AP = AQ = R$ and $\angle PAQ = 60^\circ$, so ΔAPQ is isosceles and has an angle of 60°, so must be equilateral. Therefore, $PQ = R$, or $PQ^2 = R^2$.

Now looking at ΔBPQ. Then $BP = BQ = r$, and $\angle PBQ = 90^\circ$, so $PQ^2 = r^2 + r^2 = 2r^2$, by Pythagoras.

So we know that $PQ^2 = R^2 = 2r^2$, or $\pi R^2 = 2\left(\pi r^2\right)$.

But the expression on the left side is the area of the circle with centre A and the expression in the parentheses on the right side is the area of the circle with centre B.

Thus, the required ratio is 2:1, since the area of the left-hand circle is twice the area of the right-hand circle.

ANSWER: (D)

23. *Solution 1*

Suppose that the escalator was two floors long, instead of just one, and that Jack and Jill start walking at the same time. Then Jack will reach the second floor at the same time Jill reaches the first floor (since it takes Jill twice as long to climb one floor). In that time, Jack will have climbed $2(29)$ steps and Jill will have climbed 11 steps, so there will be $47 = 2(29) - 11$ steps between them on the escalator.

These 47 steps represents the distance between two floors, or the length of the escalator.

Solution 2

Suppose that N is the total number of steps on the elevator.

As Jack walks up the escalator, he walks 29 steps, so he is carried $N - 29$ steps by the escalator.

As Jill walks up the escalator, she walks 11 steps, so is carried $N - 11$ steps by the escalator.

Since Jill's trip takes twice as long as Jack's trip, then she must be carried up twice the number of steps that Jack is carried, ie.

$$N-11=2(N-29)$$
$$N-11=2N-58$$
$$N=47$$

Therefore, the number of steps is 47.

ANSWER: (A)

24. *Solution 1*

Suppose that the artist uses squares of side length s, and that he uses M of the squares along the length, and N along the width. (Because of the constraints of the problem, M and N must both be integers.)

Then we must have $Ms = 60\frac{1}{2}$ cm and $Ns = 47\frac{2}{3}$ cm.

The total number of squares covering the rectangle is MN, and so we don't need to know s. Dividing the two equations will cancel s to obtain

$$\frac{Ms}{Ns} = \frac{60\frac{1}{2}}{47\frac{2}{3}}$$

or

$$\frac{M}{N} = \frac{\left[\frac{121}{2}\right]}{\left[\frac{143}{3}\right]} = \frac{363}{286} = \frac{11(33)}{11(26)} = \frac{33}{26}$$

So we want to determine positive integers M and N as small as possible so that $\frac{M}{N} = \frac{33}{26}$.

Since the fraction $\frac{33}{26}$ is in lowest terms, then the smallest integers M and N that will work are $M = 33$ and $N = 26$, which gives a total of $MN = (33)(26) = 858$ squares.

Solution 2

The area of the rectangle is $\frac{121}{2} \times \frac{143}{3} = \frac{11\times 11}{2} \times \frac{11\times 13}{3}$.

We would like to write this expression as the number of small squares times the area of the square, that is as an integer times the square of some number.

We can do this by rearranging the above to give

$$\frac{11\times 11}{2} \times \frac{11\times 13}{3} = \frac{11\times 13}{6} \times 11^2$$

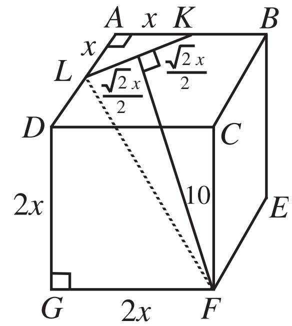
At this point, we need to make the first factor into an integer, so we want to incorporate the 6 into the "squared" part, which we can do by rewriting again as

$$\frac{11\times 13}{6} \times 11^2 = \frac{6\times 11\times 13}{6^2} \times 11^2 = [6\times 11\times 13] \times \left(\frac{11}{6}\right)^2 = 858 \times \left(\frac{11}{6}\right)^2$$

Thus there are 858 squares which are $\frac{11}{6}$ by $\frac{11}{6}$.

ANSWER: (B)

25. We label the cube as shown in the diagram.
Notice that since $AL = AK$ that ΔALK is isosceles and right-angled. This means that $LK = \sqrt{2}x$.
Next, we draw the line from F which is perpendicular to LK and meets LK at Q. By symmetry, Q is the midpoint of LK. Thus we can label $FQ = 10$ and $LQ = \frac{\sqrt{2}x}{2}$.



Using Pythagoras in ΔDGF, $DF^2 = (2x)^2 + (2x)^2 = 8x^2$.
Also, ΔFDL is right-angled at D, so using Pythagoras again, $FL^2 = 8x^2 + x^2 = 9x^2$.
Lastly, using Pythagoras in ΔLQF,

$$LF^2 = LQ^2 + QF^2$$
$$9x^2 = \left(\frac{\sqrt{2}x}{2}\right)^2 + 10^2$$
$$9x^2 = \tfrac{1}{2}x^2 + 100$$
$$\tfrac{17}{2}x^2 = 100$$
$$x \approx 3.429$$

The volume is thus $8x^3 \approx 322.82$, which rounding off is 323.

ANSWER: (A)