Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2005 Fermat Contest

(Grade 11)

Wednesday, February 23, 2005

Solutions

1. Calculating, $\dfrac{150 + (150 \div 10)}{15 - 5} = \dfrac{150 + 15}{10} = \dfrac{165}{10} = 16.5$.

Answer: (E)

2. Since $\frac{3}{9} = \frac{1}{3}$, then $\frac{1}{2} - \frac{1}{3} + \frac{3}{9} = \frac{1}{2}$.

Answer: (B)

3. *Solution 1*

Since $a = \frac{1}{2}$ and $b = \frac{2}{3}$, then $\dfrac{6a + 18b}{12a + 6b} = \dfrac{6(\frac{1}{2}) + 18(\frac{2}{3})}{12(\frac{1}{2}) + 6(\frac{2}{3})} = \dfrac{3 + 12}{6 + 4} = \dfrac{15}{10} = \dfrac{3}{2}$.

Solution 2

Since $a = \frac{1}{2}$ and $b = \frac{2}{3}$, then $\dfrac{6a + 18b}{12a + 6b} = \dfrac{6(a + 3b)}{6(2a + b)} = \dfrac{a + 3b}{2a + b} = \dfrac{\frac{1}{2} + 3(\frac{2}{3})}{2(\frac{1}{2}) + \frac{2}{3}} = \dfrac{2\frac{1}{2}}{1\frac{2}{3}} = \dfrac{\frac{5}{2}}{\frac{5}{3}} = \dfrac{3}{2}$.

Answer: (E)

4. Since $\sqrt{4 + 9 + x^2} = 7$, then $4 + 9 + x^2 = 7^2$ or $13 + x^2 = 49$ or $x^2 = 36$.

Therefore, the possible values for x are $x = \pm 6$. Therefore, the answer is (A).

Answer: (A)

5. After the coin has rolled from P to Q, the F on the face of the coin has rotated 270° clockwise. Therefore, since the distance from Q to R equals the distance from P to Q, then the F will rotate another 270° clockwise, and so the orientation of the coin will be .

Answer: (C)

6. The sequence repeats every 4 terms.

How many times will the pattern $1, 2, 3, 4$ occur in the first 2005 terms?

Since 2005 divided by 4 gives a quotient of 501 and a remainder of 1, then the first 2005 terms contain the pattern $1, 2, 3, 4$ a total of 501 times (ending at the 2004th term).

Also, the 2005th term is a 1.

Therefore, the sum of the first 2005 terms is $501(1 + 2 + 3 + 4) + 1 = 501(10) + 1 = 5011$.

Answer: (A)

7. We are told that $\angle A = \angle B + 21^\circ$ and $\angle C = \angle B + 36^\circ$.

Since the sum of the angles in a triangle is 180°, then

$$\begin{aligned} \angle A + \angle B + \angle C &= 180^\circ \\ \angle B + 21^\circ + \angle B + \angle B + 36^\circ &= 180^\circ \\ 3(\angle B) + 57^\circ &= 180^\circ \\ 3(\angle B) &= 123^\circ \\ \angle B &= 41^\circ \end{aligned}$$

Answer: (B)

8. *Solution 1*

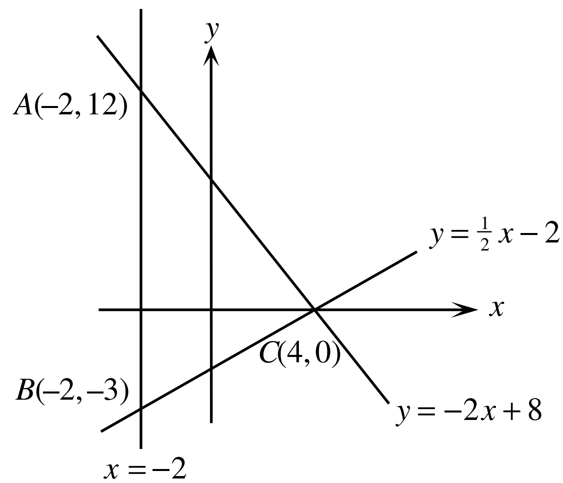
Since the seven children were born in seven consecutive years, then the oldest child is 4 years older than the oldest of the three youngest children, the second oldest child is 4 years older than the second oldest of the three youngest children, and the third oldest child is 4 years older than the youngest child.

Since the sum of the ages of the three youngest children is 42, then the sum of the ages of the three oldest children is $42 + 4 + 4 + 4 = 54$.

Solution 2
Since the ages of the seven children are seven consecutive integers, let the ages of the youngest three children be x, $x + 1$ and $x + 2$.
Then $x + x + 1 + x + 2 = 42$ or $3x + 3 = 42$ or $x = 13$.
So the ages of the seven children are 13, 14, 15, 16, 17, 18, and 19.
Therefore, the sum of the ages of the oldest three children is $17 + 18 + 19 = 54$.

ANSWER: (B)

9. We first determine where the lines $y = -2x + 8$ and $y = \frac{1}{2}x - 2$ cross the line $x = -2$.
For the line $y = -2x + 8$, when $x = -2$, $y = -2(-2) + 8 = 12$, so the point of intersection is $(-2, 12)$.
For the line $y = \frac{1}{2}x - 2$, when $x = -2$, $y = \frac{1}{2}(-2) - 2 = -3$, so the point of intersection is $(-2, -3)$.



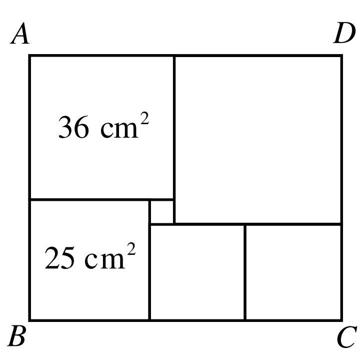
Therefore, we can think of $\triangle ABC$ as having base AB of length $12 - (-3) = 15$ and height being the distance from C to the line segment AB, or $4 - (-2) = 6$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(15)(6) = 45$.

ANSWER: (E)

10. Since 50% of P equals 20% of Q, then $\frac{1}{2}P = \frac{1}{5}Q$ or $P = \frac{2}{5}Q$.
Therefore, P is 40% of Q.

ANSWER: (C)

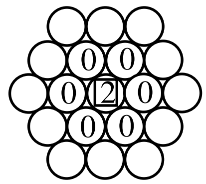
11. Since the area of the top left square is 36 cm^2, then its side length is 6 cm.
Since the area of the bottom left square is 25 cm^2, then its side length is 5 cm.
Therefore, the height AB of rectangle $ABCD$ is $5 + 6 = 11$ cm.



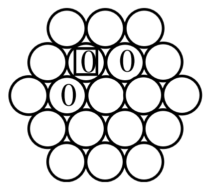
Since the side length of the bottom left square is 5 cm and the side length of the upper left square is 6 cm, then the side length of the small square in the middle must be 1 cm.
This tells us that the side length of the top right square is $6 + 1 = 7$ cm.
Therefore, the width AD of rectangle $ABCD$ is $6 + 7 = 13$ cm.
Thus, the perimeter of rectangle $ABCD$ is $2(13) + 2(11) = 48$ cm.

ANSWER: (E)

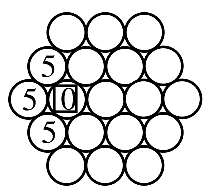
12. From the 2 in the centre, there are 6 possible 0s to which we can move.



From any 0, there are 2 possible 0s to which we can move.



From any 0, there are 3 possible 5s to which we can move.

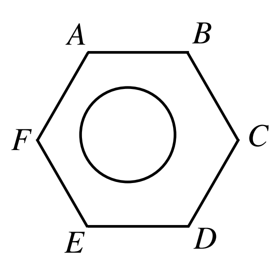


For each of the 6 choices of the first 0, we can choose either of the 2 choices for the second 0, and from whichever second 0 is chosen we can choose any of the 3 possible 5s.
Therefore, there are $6 \times 2 \times 3 = 36$ possible paths that can be followed.

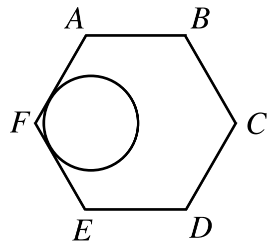
ANSWER: (A)

13. After fiddling around for a few minutes, we can see that the answer should be 2 sides. While we can convince ourselves that the answer is 2, how can we justify this mathematically? This turns out to be tricky.

Consider hexagon $ABCDEF$ and a circle contained entirely inside it.



Clearly, it is possible for the circle to touch 1 or 2 sides only.



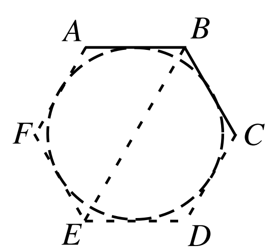
Next, we make a few notes:

- If the circle touches all 6 sides, then its radius is half the distance between opposite sides (eg. AB and DE).
- If the circle actually touches one pair of opposite sides, then its radius is half the distance between opposite sides, so in order to be contained entirely inside the hexagon, it must touch all 6 sides.
- If a circle inside the hexagon touches at least 4 sides, then it must touch at least one pair of opposite sides, so must touch all 6 sides. So if the circle touches fewer than six sides, it must touch 1, 2 or 3 sides.

Is it possible for the circle to touch 3 sides and not all 6?
If so, then we need to make sure that the circle does not touch opposite sides.
There are two ways in which this can be done – if the circle touches three consecutive sides (for example, AB, BC, CD) or three sides no two of which are consecutive (for example, AB, CD, EF).

In order to complete our justification, we need to examine these two cases and use the fact that if a circle is tangent to two lines, then its centre must lie on the angle bisector of the angle formed at the point of intersection of these two lines.

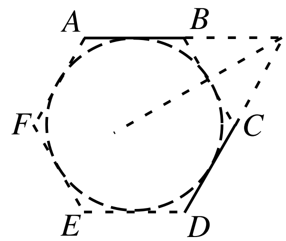
Case 1: Circle touches AB, BC, CD
Since the circle is tangent to AB and BC, then its centre must lie on the angle bisector of $\angle ABC$, which is diagonal BE of the hexagon.



Since the circle is tangent to BC and CD, then its centre must lie on the angle bisector of $\angle BCD$, which is diagonal CF of the hexagon.
Since the centre lies on BE and on CF, then the centre of the circle lies at the centre of the hexagon, and so the circle must touch all 6 sides of the hexagon.

Case 2: Circle touches AB, CD, EF

Since the circle is tangent to AB and CD, then its centre must lie on the angle bisector of the angle formed by extending AB and DC to their point of intersection. This angle bisector will be, by symmetry, the perpendicular bisector of BC.



Since the circle is tangent to CD and EF, then its centre must lie on the angle bisector of the angle formed by extending CD and EF to their point of intersection. This angle bisector will be, by symmetry, the perpendicular bisector of DE.
Since the centre lies on the perpendicular bisectors of BC and DE, then the centre of the circle lies at the centre of the hexagon, and so the circle must touch all 6 sides of the hexagon.

Therefore, if the circle touches at least 3 sides of the hexagon, then it touches all 6 sides.
Thus, the maximum number of sides it can touch without touching all 6 sides is 2 sides.

(This problem is an example of one where the answer can be obtained relatively quickly, but the justification is quite difficult. We have included this justification for completeness.)

ANSWER: (B)

14. *Solution 1*
Let the weight of the lioness be L.
Then the weight of the female cub is $\frac{1}{6}L$ and the weight of the male cub is $\frac{1}{4}L$.
Thus, $\frac{1}{4}L - \frac{1}{6}L = 14$ or $\frac{3}{12}L - \frac{2}{12}L = 14$ or $\frac{1}{12}L = 14$ or $L = 168$.
Therefore, the weight of the lioness is 168 kg.

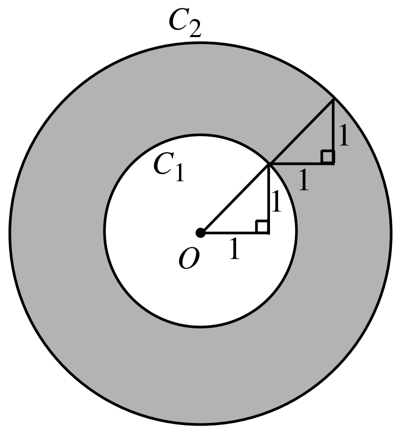
Solution 2
Let the weight of the female cub be F.
Then the weight of the male cub is $F + 14$.
Also, the weight of the lioness is equal to both of $6F$ and $4(F + 14)$.
Therefore, $6F = 4F + 56$ or $2F = 56$ or $F = 28$.
Thus, the weight of the lioness is $6(28) = 168$ kg.

ANSWER: (C)

15. Since $(x - 4)(5x + 2) = 0$, then $x - 4 = 0$ or $5x + 2 = 0$.
If $x - 4 = 0$, then $x = 4$, and so $5x + 2 = 22$.
If $5x+2 = 0$, then $5x+2 = 0$. (Here, we don't have to actually figure out x, which is $x = -\frac{2}{5}$.)
Therefore, the two possible values of $5x + 2$ are 0 and 22.

ANSWER: (C)

16. Each of the two right angled triangles has hypotenuse $\sqrt{1^2+1^2}=\sqrt{2}$, so the radius of C_1 is $\sqrt{2}$ and the radius of C_2 is $2\sqrt{2}$.



The area of the shaded region is the difference between the areas of C_2 and C_1, or

$$\pi(2\sqrt{2})^2-\pi(\sqrt{2})^2=8\pi-2\pi=6\pi$$

Answer: (D)

17. The volume of a cylinder with radius r and height h is $\pi r^2 h$.
When the cylinder with radius 2 cm and height 8 cm is full of water, it contains $\pi\times 2^2\times 8$ or 32π cm^3 of water.
When this water is poured into the second cylinder, suppose it fills this second cylinder to a depth of h.
Then $32\pi=\pi(4^2)h$ or $16\pi h=32\pi$ or $h=2$ cm.
Therefore, the depth of water in the second cylinder will be 2 cm.

Answer: (B)

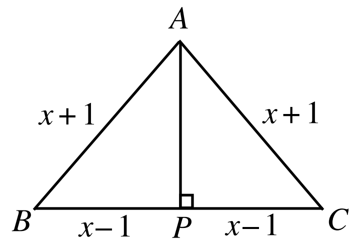
18. A score of 11 points can be obtained with 3 correct, 2 unanswered and 5 wrong.
A score of 13 points can be obtained with 4 correct, 1 unanswered and 5 wrong.
A score of 17 points can be obtained with 5 correct, 2 unanswered and 3 wrong.
A score of 23 points can be obtained with 7 correct, 2 unanswered and 1 wrong.
Therefore, by process of elimination, 29 is the total score which is not possible.
(Why is 29 not possible? If all ten questions are correct, the total score would be 30 points. If 9 or fewer questions are correct, at least 2 points will be lost from the maximum possible, ie. the maximum possible score is 28. Therefore, 29 is not possible.)

Answer: (E)

19. Since Chris bicycles at 24 km/h and Sam bicycles at 16 km/h, then Chris gains 8 km/h on Sam.
Since Sam starts 1 km north of Chris, then it takes Chris $\frac{1}{8}$ of an hour, or $\frac{1}{8}\times 60=\frac{60}{8}=\frac{15}{2}$ or $7\frac{1}{2}$ minutes to catch Sam.

Answer: (D)

20. Draw the altitude from A to P on BC.
Since $\triangle ABC$ is isosceles, then P is the midpoint of BC, so $BP = PC = x - 1$.



By the Pythagorean Theorem,

$$AP = \sqrt{AB^2 - BP^2} = \sqrt{(x+1)^2 - (x-1)^2} = \sqrt{(x^2 + 2x + 1) - (x^2 - 2x + 1)} = \sqrt{4x} = 2\sqrt{x}$$

Therefore, the area of $\triangle ABC$ is equal to

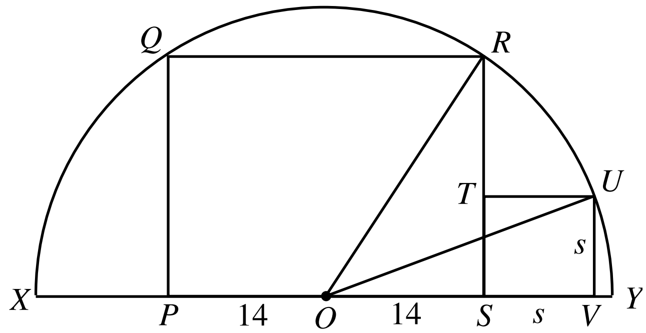
$$\frac{1}{2}(BC)(AP) = \frac{1}{2}(2x - 2)(2\sqrt{x}) = 2(x - 1)\sqrt{x}$$

Answer: (E)

21. Consider a^b and choose a and b to be two different numbers from -1, -2, -3, -4, and -5.
What is the largest possible value for a^b?
Since b will be negative, we write $a^b = \dfrac{1}{a^{-b}}$ and here $-b > 0$.
If b is odd, then since a is negative, a^b will be negative.
If b is even, then a^b will be positive.
So to make a^b as big as possible, we make b even (ie. -2 or -4).
Also, in order to make $a^b = \dfrac{1}{a^{-b}}$ as big as possible, we want to make a^{-b} as small as possible, so a should be as small as possible in absolute value.
Therefore, the largest possible value of a^b will be when $a = -1$ and b is either -2 or -4, giving in either case 1 (ie. $(-1)^{-2} = (-1)^{-4} = 1$).
What is the second largest possible value for a^b?
Again, we need b to be even to make a^b positive, and here we can assume that $a \neq -1$.
To make a^b as large as possible, then using similar logic, we choose $b = -2$ and $a = -3$, giving $a^b = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$.
Therefore, the two largest possible values for a^b are 1 and $\frac{1}{9}$.
Thus, looking at $a^b + c^d$, since -1 can only be chosen for one of these four numbers, then the largest possible value for this expression is the sum of the largest two possible values for a^b, ie. $1 + \frac{1}{9} = \frac{10}{9}$, which is obtained by calculating $(-1)^{-4} + (-3)^{-2}$.

Answer: (D)

22. Let O be the centre of the circle, let r be the radius of the circle, and let s be the side length of the square. We want to calculate s^2.
By symmetry, O is the midpoint of PS, so $OP = OS = \frac{1}{2}QR = 14$.
Join O to R and O to U. We see that $OR = OU = r$, the radius of the circle.



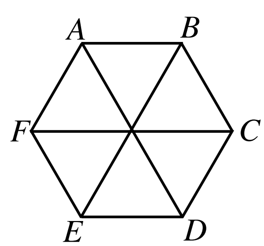
Since $\triangle OSR$ is right-angled at S, then $OR^2 = OS^2 + SR^2$ by the Pythagorean Theorem, or $r^2 = 14^2 + 12^2 = 196 + 144 = 340$.
Since $\triangle OVU$ is right-angled at S, then $OU^2 = OV^2 + VU^2$ or $r^2 = (14+s)^2 + s^2$.
But $r^2 = 340$, so

$$\begin{aligned} 340 &= s^2 + 28s + 196 + s^2 \\ 0 &= 2s^2 + 28s - 144 \\ 0 &= s^2 + 14s - 72 \\ 0 &= (s+18)(s-4) \end{aligned}$$

Since s must be positive, then $s = 4$, so the area of $STUV$ is $s^2 = 16$.

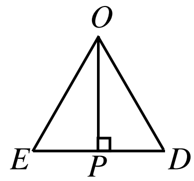
Answer: (C)

23. When the cube is sliced in half in this manner, each half cube will have 7 faces: one hexagonal face from the slice, and 6 faces each of which is part of a face of the original cube.
By symmetry, the total area of these last 6 faces will be half of the surface area of the original cube, or $\frac{1}{2} \times 6 \times 4^2 = 48$ cm^2.
Lastly, we must calculate the area of the hexagonal face. By symmetry, the face is a regular hexagon. The side length is equal to the length of the line segment joining the midpoints of adjacent sides of a square of side length 4, or $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.
So we need to calculate the area of a regular hexagon of side length $2\sqrt{2}$.
Consider a regular hexagon $ABCDEF$. Each interior angle is 120°.
Draw diagonals AD, BE and CF.



By symmetry, these diagonals divide the hexagon into 6 equilateral triangles, each of side length $2\sqrt{2}$.

Consider one of these equilateral triangles.
Draw altitude from vertex O to P on side DE.



Since $\triangle ODE$ is equilateral, P is the midpoint of DE, so $EP = \sqrt{2}$.
But $\triangle OPE$ is 30°-60°-90°, so $OP = \sqrt{3}EP = \sqrt{6}$.
Therefore, the area of $\triangle ODE = \frac{1}{2}(ED)(OP) = \frac{1}{2}(2\sqrt{2})(\sqrt{6}) = \sqrt{12} = 2\sqrt{3}$ cm^2.
Therefore, the area of hexagon $ABCDEF$ is $6(2\sqrt{3}) = 12\sqrt{3}$ cm^2.
Therefore, the total surface area of each half cube is $48 + 12\sqrt{3} \approx 69$ cm^2.

Answer: (A)

24. When we start at the first term and look at every other term from there, we will look at the last term. Therefore, the total number of terms after the first is a multiple of 2.
When we start at the first term and look at every third term from there, we will look at the last term. Therefore, the total number of terms after the first is a multiple of 3.
Thus, the total number of terms after the first must be a multiple of 6, so the total number of terms in the sequence can be written as $6k + 1$.
Now we look at the given sums, knowing that $n = 6k + 1$.
When we add up every other term, including the first and the last, we are adding up $3k + 1$ terms in total.
Therefore, $\frac{1}{2}(3k + 1)(a + a + 6kd) = 320$ or $(3k + 1)(2a + 6kd) = 640$.
(The sum of an arithmetic sequence is half of the product of the number of terms and the sum of the first and last terms. The sequence we get by looking at every other term (or every third term) of an arithmetic sequence is again arithmetic.)
When we add up every third term, including the first and the last, we are adding up $2k + 1$ terms in total.
Therefore, $\frac{1}{2}(2k + 1)(a + a + 6kd) = 224$, or $(2k + 1)(2a + 6kd) = 448$.
Dividing the first equation by the second, we obtain $\dfrac{3k + 1}{2k + 1} = \dfrac{640}{448} = \dfrac{10}{7}$, so $k = 3$.
Thus, $(3(3) + 1)(2a + 6kd) = 640$ or $2a + 6kd = 64$.
We want to determine the sum of the entire arithmetic sequence, which is

$$\frac{1}{2}(6k + 1)(a + a + 6kd) = \frac{1}{2}(19)(2a + 6kd) = \frac{1}{2}(19)(64) = 608$$

Answer: (C)

25. Unfortunately, there was a problem with this question that we did not discover until after the Contests had been written. Our thanks go to Dr. Yongmoo Kim for pointing this out.

If the problem had been posed as

> A *triline* is a line with the property that three times its slope is equal to the sum of its
> x-intercept and its y-intercept. For how many integers q with $1 \leq q \leq 10\,000$ is there at least one <u>positive</u> integer p so that there is exactly one triline through (p, q)?

(ie. with the underlined word positive added), then the following solution would have been correct:

Consider a line through the point (p, q) with slope m.
The equation of this line is $y = m(x - p) + q = mx + (q - mp)$.
Thus, the y-intercept of this line is $y = q - mp$ and the x-intercept comes from setting $y = 0$, which gives $x = \dfrac{mp - q}{m}$.
For this line to be a triline, we need $3m = (q - mp) + \dfrac{mp - q}{m}$ or $3m^2 = qm - pm^2 + mp - q$ or $(3 + p)m^2 - (p + q)m + q = 0$.
Given a fixed point (p, q), for there to be only one triline through (p, q), there can be only one slope m satisfying $(3 + p)m^2 - (p + q)m + q = 0$, ie. this quadratic equation has exactly one real root. (The leading coefficient $3 + p$ is non-zero since p is positive.)
So for a fixed point (p, q), the condition that there be only one triline through (p, q) is that the discriminant of $(3 + p)m^2 - (p + q)m + q = 0$ equals 0, or

$$\begin{aligned}
(p + q)^2 - 4(3 + p)q &= 0 \\
p^2 + 2pq + q^2 - 12q - 4pq &= 0 \\
p^2 - 2pq + q^2 - 12q &= 0 \\
(p - q)^2 &= 12q
\end{aligned}$$

So we must determine the number of integers q with $1 \leq q \leq 10\,000$ such that there is an integer p such that $(p - q)^2 = 12q$.
In order for this to be true, $12q$ needs to be a perfect square, so $3q$ needs to be a perfect square.
In order for $3q$ to be a perfect square, q needs to be 3 times a perfect square (since q must contain an odd number of factors of 3 and even number of every other prime factor).
If $q = 3k^2$, then we can solve $(p - q)^2 = 12q$ since then $(p - 3k^2)^2 = 36k^2$ or $p - 3k^2 = \pm 6k$ or $p = 3k^2 \pm 6k$.
So how many integers q between 1 and 10 000 are of the form $q = 3k^2$? The minimum value of k that works is $k = 1$ and the maximum is $k = 57$ (since $3(58)^2 = 10\,092$ is too large).
Therefore, there are 57 such values of q, ie. the answer would be (B).

However, the problem was posed as

> A *triline* is a line with the property that three times its slope is equal to the sum of its
> x-intercept and its y-intercept. For how many integers q with $1 \leq q \leq 10\,000$ is there at least one integer p so that there is exactly one triline through (p, q)?

Following the above solution, we arrive at needing one slope m to satisfy the equation

$$(3+p)m^2 - (p+q)m + q = 0$$

There are two ways for this "quadratic" equation to have a single root – either if the discriminant is 0, or if the leading coefficient $p+3$ is 0 (ie. the "quadratic" is actually linear).
So if $p=-3$, then we need the equation $(q-3)m+q=0$ to have exactly one solution for m, which it does *for every* q, as long as $q \neq 3$.
If $q=3$, then the value of $p=9$ makes the discriminant of the quadratic equation 0.
In other words, every value of q between 1 and 10 000 has at least one integer p so that there is exactly one triline through (p,q), so there are 10 000 values of q that work.

Our apologies for any confusion that this may have caused.