



**Canadian
Mathematics
Competition**

*An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario*

2005 Galois Contest

Wednesday, April 20, 2005

Solutions

1. (a) *Solution 1*

We list the first 11 terms, obtaining each by adding 5 to the previous term:

$$17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67$$

Therefore, the 11th term is 67.

Solution 2

Since the constant that we add to each term to obtain the next is 5, then the 11th term is $17 + 5(10) = 67$, since we must add the common difference 10 times to the 1st term to obtain the 11th term.

(b) *Solution 1*

Every term in the first sequence ends in a 7 or a 2. This is because the difference between consecutive terms is always 5, and whenever we add 5 to a number ending in 7, we obtain a number ending in 2, and vice versa.

Every term in the second sequence ends in a 3 or an 8. This is because the difference between consecutive terms is always 15, and whenever we add 15 to a number ending in 3, we obtain a number ending in 8, and vice versa.

No number can occur in both sequences, since no number can end in two different digits.

Solution 2

Every number in the first sequence is of the form $17 + 5n$ for some non-negative integer n . Every number in the second sequence is of the form $13 + 15m$ for some non-negative integer m .

For a number to occur in both sequences, we would need $17 + 5n = 13 + 15m$ or $4 = 15m - 5n = 5(3m - n)$.

But the right side is a multiple of 5, and the left side is not, so this is impossible.

Therefore, there is no number which occurs in both sequences.

(c) *Solution 1*

We see first that 22 occurs in both sequences.

Since the difference between consecutive terms is 5 in the first sequence and 6 in the second sequence, then a number $5 \times 6 = 30$ greater than a number which occurs in both sequences will again occur in both sequences. (This is because adding 30 to a term in the first sequence moves us six terms along the sequence and adding 30 to a term in the second sequence moves us five terms along.)

Therefore, terms which occur in both sequences are 22, 52, 82, 112, 142, 172, 202, 232, 262, 292, 322, 352, 382, and 412.

Thus, 412 is a number between 400 and 420 which occurs in both sequences.

Solution 2

Since the difference between consecutive terms in the first sequence is 5, then the first sequence contains every positive integer which is at least 17 and ends in a 2 or a 7.

Therefore, the numbers in the first sequence which are between 400 and 420 are 402, 407, 412 and 417.

Which of these four possibilities occurs in the second sequence?

Since the difference between consecutive terms in the second sequence is 6, then every number in the second sequence is of the form $16 + 6n$, where n is a non-negative integer. Does $16 + 6n = 402$ have a solution where n is an integer? This would mean $6n = 386$, but 386 is not divisible by 6, so there is no solution.

Does $16 + 6n = 407$ have a solution where n is an integer? No, since the left side is even and the right side is odd.

Does $16 + 6n = 412$ have a solution where n is an integer? Yes, since if $16 + 6n = 412$ then $6n = 396$ or $n = 66$.

Therefore, 412 occurs in both sequences.

(We could also check that 417 cannot occur in the second sequence, since it is odd.)

2. (a) The tiles which remain to be placed are one 5 and two 6s. There are only 3 different ways in which these tiles can be placed. Here are the three ways along with the totals of each column and row:

1	3	6	5
3	1	2	2
5	4	4	6

(Column totals: 9, 8, 12, 13; Row totals: 15, 8, 19)

1	3	5	5
3	1	2	2
6	4	4	6

(Column totals: 10, 8, 11, 13; Row totals: 14, 8, 20)

1	3	6	5
3	1	2	2
6	4	4	5

(Column totals: 10, 8, 12, 12; Row totals: 15, 8, 19)

Of these the ones which give Omar more points than Emilia is the first

1	3	6	5
3	1	2	2
5	4	4	6

where Omar gets 4 points and Emilia gets 3.

- (b) The total number of points available is 7, since there are 4 columns and 3 rows and each column and each row gives 1 point to one of the players.
So Emilia's score and Omar's score must add up to 7.
Since 7 is odd, Emilia's score and Omar's score cannot be the same.

- (c) *Solution 1*

The tiles that remain to be placed are 2, 2, 4 and 5, ie. three even numbers and one odd number.

In the grid, the fourth column is already complete and has an even sum (16), so Emilia already has one point.

Consider rows 1 and 3 and columns 1 and 3.

Each has only one space left open, and each has an even sum so far.

In any of these cases, if an even number is placed in the empty space, the row or column will be complete and the sum will be even, giving Emilia a point. If an odd number is placed in the empty space, the row or column will be complete and the sum will be odd, giving Omar a point.

But since there is only one odd-numbered tile left to play, then only one of these four rows and columns can end up with an odd score, and three are guaranteed to be even.

So no matter where Omar places the 5, Emilia will get 3 more points, for a total of at least 4 points.

Since there are only 7 points available, then Emilia is guaranteed to have more points than Omar, because Omar will get at most 3 points.

Solution 2

The tiles that remain to be placed are 2, 2, 4 and 5, ie. three even numbers and one odd number.

Suppose Omar places the 5 in the leftmost empty box. Then the numbers which go into the other three empty spaces are the 2, 2 and 4, so are all even.

The sum of the top row will then be $1 + 3 + 6$ plus either 2 or 4, so will be even.

The sum of the bottom row will be $3 + 1 + 6$ plus either 2 or 4, so will be even.

The sum of the fourth column is already 16, so is even.

The sum of the third column will be $3 + 1$ plus either 2 or 4, so will be even.

Therefore, in this case Emilia will get at least 4 of the 7 possible points, so will have more points than Omar.

If Omar places the 5 in the rightmost empty box, the argument is exactly the same, except we look at the first column instead of the third. In this case, again Emilia gets at least 4 points.

Suppose Omar places the 5 in the uppermost empty box. Then the numbers which go into the other three empty spaces are the 2, 2, and 4, so are all even.

The sum of the bottom row will be $3 + 1 + 6$ plus either 2 or 4, so will be even.

The sum of the fourth column is already 16, so is even.

The sum of each of the first and third columns is $3 + 1$ plus either 2 or 4, so will be even.

Therefore, in this case Emilia will get at least 4 of the 7 possible points, so will have more points than Omar.

If Omar places the 5 in the lowermost empty box, the argument is exactly the same as this previous case, except we look at the top row instead of the bottom row. In this case, again Emilia gets at least 4 points.

Therefore, having looked at all of the possible cases, Emilia will always get at least 4 points, so will always have more points than Omar, no matter where Omar places the 5.

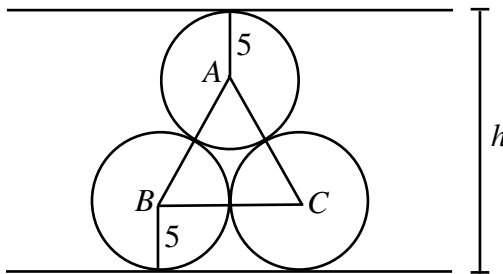
3. (a) In Crate A, there are 10 pipes in each row. Since there are 200 pipes to pack, there will be 20 rows in total.

In Crate B, there are 10 pipes in every other row and 9 pipes in the in-between rows. Thus, there are 19 pipes in every pair of neighbouring rows. So with 20 rows, there will be 190 pipes, and the topmost row will have 9 pipes in it, since even-numbered rows contain 9 pipes. Thus, if we add one more row, we will add 10 more pipes, for 200 in total. So in Crate B, there will be 21 rows in total.

- (b) Join the centres A , B and C of the three circles.

The lines AB , BC and CA will pass through the points where the circles touch, so will each have length 10 cm (that is, twice the radius of one of the circles).

We can break the height of the pile into three pieces: the distance from the bottom of the pile to the line BC , the height of the equilateral triangle ABC , and the distance A to the top of the pile.



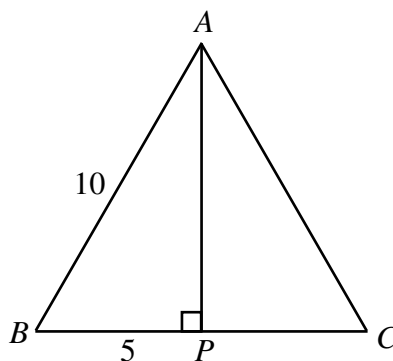
The first and last of these distances are each equal to the radius of one of the circles, that is, 5 cm.

So we must determine the height of $\triangle ABC$, which is an equilateral triangle with side length 10 cm.

There are many ways to do this.

Drop a perpendicular from A to P on BC .

Since $AB = AC$, then P is the midpoint of BC , so $BP = 5$ cm.

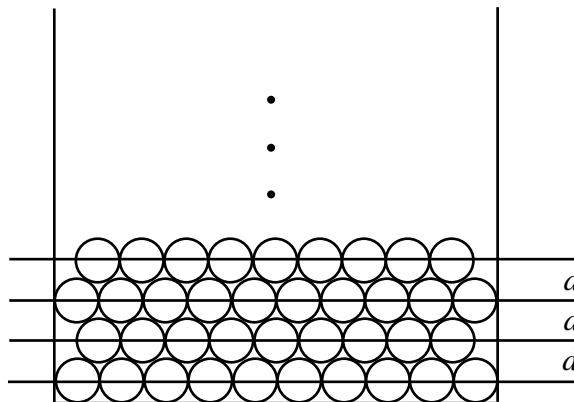


Then $\triangle ABP$ is a 30° - 60° - 90° triangle, so $AP = \sqrt{3}BP = 5\sqrt{3}$ cm.

Thus, the height of the pile is $5 + 5\sqrt{3} + 5 = (10 + 5\sqrt{3})$ cm.

- (c) In Crate A, we have 20 rows of 10 pipes packed directly on top of each other. So the height of the packing is 20 times the diameter of a single pipe, or 200 cm.

In Crate B, draw a horizontal line through the centres of the 9 or 10 pipes in each row. By symmetry, the distance between each consecutive pair of these 21 lines will be the same, say equal to d . There will be 20 such distances.



The distance of the bottom line from the bottom of the crate is equal to the radius of a pipe, and the distance of the top line from the top of the top row is also equal to the radius of a pipe.

Thus, the total height of the packing in Crate B is equal to $(10 + 20d)$ cm.

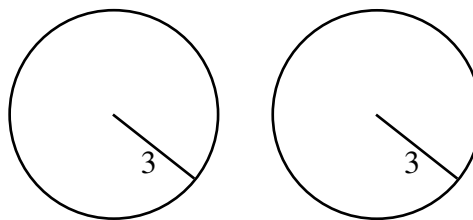
What is d equal to?

If we extract three pairwise touching pipes from two consecutive rows, we get a pile exactly as in (b), so d is equal to the height of the equilateral triangle whose height we calculated in (b), ie. $d = 5\sqrt{3}$ cm.

Therefore, the total height of this packing is $(10 + 100\sqrt{3})$ cm, which is approximately 183.2 cm.

Therefore, the difference in the total heights of the two packings is $200 - (10 + 100\sqrt{3}) = 190 - 100\sqrt{3}$ cm, or about 16.8 cm, with the packing in Crate A being the higher one.

4. (a) To calculate the total surface area of the cylinder, we cut off the two ends to obtain two circles of radius 3.

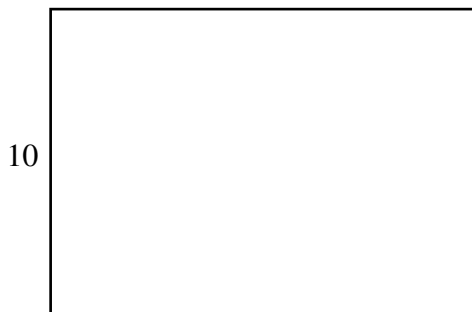


The two ends combined have an area of $2\pi r^2 = 2\pi(3^2) = 18\pi$.

Next, we must calculate the lateral surface area.

To do this, we make a vertical cut through this area, and unroll the lateral surface.

When we do this, we obtain a rectangle with height 10. The width of the rectangle (ie. the length of the top edge) is equal to the circumference of one of the ends, since the top edge of this rectangle lay exactly along the circumference of the top end.



The circumference of one of the ends is $2\pi r = 2\pi(3) = 6\pi$, so the width of the rectangle is 6π .

Therefore, the area of this rectangle is $10 \times 6\pi = 60\pi$.

So the total surface area of the cylinder is $18\pi + 60\pi = 78\pi$.

The volume of the cylinder is equal to $\pi r^2 h = \pi(3^2)(10) = 90\pi$.

- (b) Since the volume of the sphere is $\frac{4}{3}\pi r^3$ and the volume of the cylinder is $\pi r^2 H$ and these are equal, then $\frac{4}{3}\pi r^3 = \pi r^2 H$ or $H = \frac{4}{3}r$ (cancelling the common factor of πr^2).

Looking at our work from (a) more generally, we can see that the total surface area of a cylinder of radius r and height H is $2\pi r^2 + 2\pi r H$ (area of the two ends plus lateral area). We are told that the surface area of a cone with radius r , height h and slant height s is $\pi r^2 + \pi r s$.

Since the total surface areas of the cone and cylinder are equal, then

$$\begin{aligned} 2\pi r^2 + 2\pi r H &= \pi r^2 + \pi r s \\ 2r + 2H &= r + s \quad (\text{cancelling the common factor of } \pi r) \\ r + 2H &= s \end{aligned}$$

Since we want to determine some information about h and H , we should try to eliminate the s .

Look at the cone. Join the tip of the cone to the centre of the circular base. This line will be perpendicular to the base, so will form a right-angled triangle with legs r and h and hypotenuse s .

Thus, by the Pythagorean Theorem, $s = \sqrt{r^2 + h^2}$.

Therefore, $r + 2H = \sqrt{r^2 + h^2}$.

We can now use the fact that $H = \frac{4}{3}r$ or $r = \frac{3}{4}H$ to write this equation in terms of H and h only:

$$\begin{aligned}r + 2H &= \sqrt{r^2 + h^2} \\ \frac{3}{4}H + 2H &= \sqrt{\left(\frac{3}{4}H\right)^2 + h^2} \\ \frac{11}{4}H &= \sqrt{\frac{9}{16}H^2 + h^2} \\ \frac{121}{16}H^2 &= \frac{9}{16}H^2 + h^2 && \text{(squaring both sides)} \\ \frac{112}{16}H^2 &= h^2 \\ 7H^2 &= h^2\end{aligned}$$

Since $h^2 = 7H^2$, then $h = \sqrt{7}H$ and so we cannot have both h and H integers, since $\sqrt{7}$ is irrational.