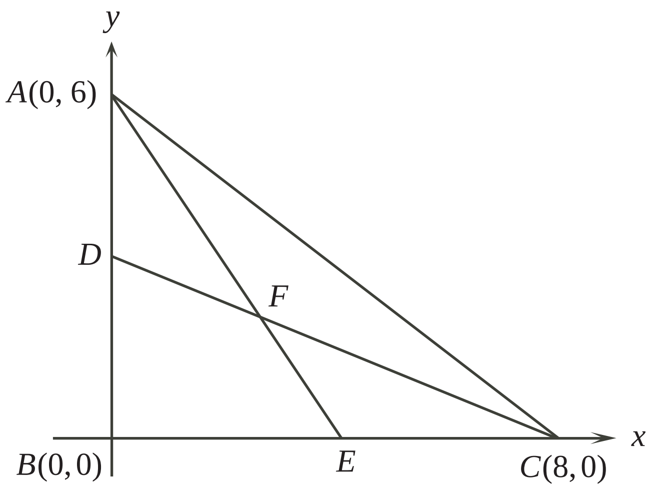
2011 Hypatia Contest (Grade 11)

Wednesday, April 13, 2011

1. In the diagram, D and E are the midpoints of AB and BC respectively.



   (a) Determine an equation of the line passing through the points C and D.

   (b) Determine the coordinates of F, the point of intersection of AE and CD.

   (c) Determine the area of $\triangle DBC$.

   (d) Determine the area of quadrilateral $DBEF$.

2. A set S consists of all two-digit numbers such that:

   - no number contains a digit of 0 or 9, and
   - no number is a multiple of 11.

   (a) Determine how many numbers in S have a 3 as their tens digit.

   (b) Determine how many numbers in S have an 8 as their ones digit.

   (c) Determine how many numbers are in S.

   (d) Determine the sum of all the numbers in S.

3. Positive integers (x, y, z) form a *Trenti-triple* if $3x = 5y = 2z$.

   (a) Determine the values of y and z in the Trenti-triple $(50, y, z)$.

   (b) Show that for every Trenti-triple (x, y, z), y must be divisible by 6.

   (c) Show that for every Trenti-triple (x, y, z), the product xyz must be divisible by 900.

4. Let $F(n)$ represent the number of ways that a positive integer n can be written as the sum of positive odd integers. For example,

- $F(5) = 3$ since

$$\begin{aligned} 5 &= 1+1+1+1+1 \\ &= 1+1+3 \\ &= 5 \end{aligned}$$

- $F(6) = 4$ since

$$\begin{aligned} 6 &= 1+1+1+1+1+1 \\ &= 1+1+1+3 \\ &= 3+3 \\ &= 1+5 \end{aligned}$$

(a) Find $F(8)$ and list all the ways that 8 can be written as the sum of positive odd integers.

(b) Prove that $F(n+1) > F(n)$ for all integers $n > 3$.

(c) Prove that $F(2n) > 2F(n)$ for all integers $n > 3$.