2020 Fryer Contest

Wednesday, April 15, 2020
(in North America and South America)

Thursday, April 16, 2020
(outside of North America and South America)

Solutions
1. (a) Locating point $C$ on the graph, Cao pays $7.00 for 14 posters which is a price of $\frac{7}{14} = 0.50$ per poster.

(b) From part (a), Daniel pays $1.60 per poster and Cao pays $0.50 per poster.
Calculating, Annie pays $10.00 for 5 posters or $\frac{10}{5} = 2.00$ per poster, Bogdan pays $\frac{8.00}{8} = 1.00$ per poster, and Emily pays $\frac{15.00}{15} = 1.00$ per poster.
Thus, Bogdan and Emily are paying the same price per poster.
(Alternatively, we may have noticed that the line from the origin $(0,0)$ passing through $E$ also passes through $B$. The slope of this line represents the price in dollars per poster for Bogdan and for Emily.)

(c) From part (a), Daniel paid the company that printed his first batch $1.60 per poster.
To print his second batch at the local library, Daniel would pay $60.00 ÷ 40 = 1.50$ per poster.
To spend less money, Daniel should print his second batch at the library.
Alternatively, Daniel paid $16.00 for his first batch of 10 posters and thus would pay $4 \times 16.00 = 64.00$ to print his second batch of 40 posters using the same company that printed his first batch.
Since the cost to print the 40 posters at the library is $60.00, he should print them at the library to save money.

(d) In part (b), we calculated that Emily’s printing company charges her $1.00 per poster.
Since this is a fixed price per poster, then Emily paid $1.00 per poster for each of the 25 posters that she had printed, for a total of $25.00 spent.
Annie and Emily each printed 25 posters and spent the same amount of money, $25.00.
Annie paid $10.00 to print her first 5 posters, and so she spent $25.00 − $10.00 = $15.00 to have her additional 20 posters printed.
Thus, Annie was charged $\frac{15.00}{20} = 0.75$ per additional poster.

2. (a) In $\triangle KLR$, we have $\angle KLR = 90^\circ$ and by using the Pythagorean Theorem, we get $LR^2 = 50^2 - 40^2 = 900$ and so $LR = \sqrt{900} = 30$ m (since $LR > 0$).

(b) We begin by showing that $\triangle JMQ$ is congruent to $\triangle KLR$.
Since $JKLM$ is a rectangle, then $JM = KL = 40$ m.
In addition, hypotenuse $JQ$ has the same length as hypotenuse $KR$, and so $\triangle JMQ$ is congruent to $\triangle KLR$ by HS congruence.
Thus, $MQ = LR = 30$ m and so $ML = 66 - 30 - 30 = 6$ m.

(c) Since $PJ = PK = 5$ m, $\triangle PJK$ is isosceles and so the height, $PS$, drawn from $P$ to $JK$ bisects $JK$, as shown.
Since $JKLM$ is a rectangle, then $JK = ML = 6$ m and so $SK = \frac{JK}{2} = 3$ m.
Using the Pythagorean Theorem in $\triangle PSK$, we get $PS^2 = 5^2 - 3^2 = 16$ and so $PS = 4$ m (since $PS > 0$).
Thus the height of $\triangle PJK$ drawn from $P$ to $JK$ is 4 m.
(d) We begin by determining the area of \( \triangle PQR \).
Construct the height of \( \triangle PQR \) drawn from \( P \) to \( T \) on \( QR \), as shown.
Since \( PT \) is perpendicular to \( QR \), then \( PT \) is parallel to \( KL \) (since \( KL \) is also perpendicular to \( QR \)).
By symmetry, \( PT \) passes through \( S \), and so the height \( PT \) is equal to \( PS + ST = PS + KL \) or \( 4 + 40 = 44 \) m.
The area of \( \triangle PQR \) is \( \frac{1}{2} \times QR \times PT = \frac{1}{2} \times 66 \times 44 = 1452 \) m
The area of \( JKLM \) is \( ML \times KL = 6 \times 40 = 240 \) m
The fraction of the area of \( \triangle PQR \) that is covered by rectangle \( JKLM \) is \( \frac{240}{1452} = \frac{20}{121} \).

3. (a) If the 5th term in a Dlin sequence is 142, then the 6th term is \((142 + 1) \times 2 = 143 \times 2 = 286\).
To determine the 4th term in the sequence given the 5th, we “undo” adding 1 followed by
doubling the result by first dividing the 5th term by 2 and then subtracting 1 from the result.
To see this, consider that if two consecutive terms in a Dlin sequence are \( a \) followed by \( b \),
then \( b = (a + 1) \times 2 \).
To determine the operations needed to find \( a \) given \( b \) (that is, to move backward in the sequence), we rearrange this equation to solve for \( a \).
\[
\begin{align*}
b &= (a + 1) \times 2 \\
\frac{b}{2} &= a + 1 \\
\frac{b}{2} - 1 &= a
\end{align*}
\]
Thus if the 5th term in the sequence is 142, then the 4th term is \( \frac{142}{2} - 1 = 71 - 1 = 70 \).
(We may check that the term following 70 is indeed \((70 + 1) \times 2 = 142\).)
(b) If the 1st term is 1406, then clearly this is a Dlin sequence that includes 1406.
If the 2nd term in a Dlin sequence is 1406, then the 1st term in the sequence is \( \frac{1406}{2} - 1 = 703 - 1 = 702 \).
If the 3rd term in a Dlin sequence is 1406, then the 2nd term is 702 (as calculated in the
line above) and the 1st term in the sequence is \( \frac{702}{2} - 1 = 351 - 1 = 350 \).
If the 4th term in a Dlin sequence is 1406, then the 3rd term is 702, the 2nd term is 350,
and the 1st term in the sequence is \( \frac{350}{2} - 1 = 175 - 1 = 174 \).
At this point, we see that 174, 350, 702, and 1406 are possible 1st terms which give a Dlin
sequence that includes 1406.
We may continue this process of working backward (dividing by 2 and subtracting 1) to
determine all possible 1st terms which give a Dlin sequence that includes 1406.
\[
1406 \rightarrow 702 \rightarrow 350 \rightarrow 174 \rightarrow \frac{174}{2} - 1 = 86 \rightarrow \frac{86}{2} - 1 = 42 \rightarrow \frac{42}{2} - 1 = 20 \rightarrow \frac{20}{2} - 1 = 9
\]
Attempting to continue the process beyond 9 gives \( \frac{9}{2} - 1 = \frac{7}{2} \) which is not possible since the
1st term in a Dlin sequence must be a positive integer (and so all terms are positive
integers).
Thus, the possible 1st terms which give a Dlin sequence that includes 1406 are 9, 20, 42,
86, 174, 350, 702, and 1406.
(c) Each of the integers from 10 to 19 inclusive is a possible first term, and so we must
determine the ones digit of each term which follows each of these ten possible first terms.
If the 1st term is 10, then the 2nd term \((10 + 1) \times 2 = 22\) has ones digit 2, and the 3rd term
(22 + 1) × 2 = 46 has ones digit 6.
If the 1st term is 11, then the ones digit of the 2nd term (11 + 1) × 2 = 24 is 4, and the 3rd term (24 + 1) × 2 = 50 has ones digit 0.
Given each of the possible first terms, we list the ones digits of the 2nd and 3rd terms in the table below.

<table>
<thead>
<tr>
<th>1st term</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units digit of the 2nd term</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Units digit of the 3rd term</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>2</td>
<td>6</td>
<td>0</td>
<td>4</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

From the table above, we see that the only ones digit which repeats itself is 8.
Thus, if the 1st term in the sequence is 18 (has ones digit 8), then the 2nd and 3rd terms in the sequence have ones digit 8 and so all terms will have the same ones digit, 8.
Similarly, if the 1st term in the sequence is 13 (has ones digit 3), then the 2nd and 3rd terms in the sequence have ones digit 8.
It then follows that all further terms after the first will have ones digit 8.
The 1st terms (from 10 to 19 inclusive) which produce a Dlin sequence in which all terms after the 1st term have the same ones digit are 13 and 18.

(d) If the 1st term in a Dlin sequence is $x$, then the 2nd term is $(x + 1) × 2 = 2x + 2$, and the 3rd term is $(2x + 2 + 1) × 2 = (2x + 3) × 2 = 4x + 6$.
For example, if $x = 1$ (note that this is the smallest possible 1st term of a Dlin sequence), then the 3rd term is $4 × 1 + 6 = 10$, and if $x = 2$, the 3rd term is $4 × 2 + 6 = 14$.
What is the largest possible value of $x$ (the 1st term of the sequence) which makes $4x + 6$ (the 3rd term of the sequence) less than or equal to 2020?
Setting $4x + 6$ equal to 2020 and solving, we get $4x = 2014$ and so $x = 503.5$.
Since the 1st term of the sequence must be a positive integer, the 3rd term cannot be 2020.
Similarly, solving $4x + 6 = 2019$, we get that $x$ is not an integer and so the 3rd term of a Dlin sequence cannot equal 2019.
When $4x + 6 = 2018$, we get $4x = 2012$ and so $x = 503$.
Thus, if a Dlin sequence has 1st term equal to 503, then the 3rd term of the sequence is a positive integer between 1 and 2020, namely 2018.
Further, 503 is the largest possible 1st term for which the 3rd term has this property.
Each 1st term $x$ will give a different 3rd term, $4x + 6$.
Thus, to count the number of positive integers between 1 and 2020, inclusive, that can be the 3rd term in a Dlin sequence, we may count the number of 1st terms which give a 3rd term having this property.
The smallest possible 1st term is 1 (giving a 3rd term of 10) and the largest possible 1st term is 503 (which gives a 3rd term of 2018).
Further, every value of $x$ between 1 and 503 gives a different 3rd term between 10 and 2018.
Thus, there are 503 positive integers between 1 and 2020, inclusive, that can be the 3rd term in a Dlin sequence.

4. (a) There are 10 different ways to colour a 5 × 1 grid so that exactly 3 cells are red and 2 cells are blue.
One way to see this is to count the number of different ways to colour 2 cells blue, since each of the 3 remaining cells must be coloured red.
(Alternatively, we could count the number of different ways to colour 3 cells red.)
There are 5 cells that may be chosen first to be coloured blue.
After this chosen cell is coloured, there are 4 remaining cells that can be chosen to be
coloured blue. Thus, there are $5 \times 4 = 20$ such choices of cells to be coloured blue. However, since the two blue cells are indistinguishable from one another, we have counted each of the different ways of colouring 2 cells blue twice.

For example, choosing to first colour the cell in the second row blue and then colouring the cell in the fifth row blue gives the same colouring as first choosing to colour the cell in the fifth row blue and then colouring the cell in the second row blue.

Thus, there are $20 \div 2 = 10$ different ways that a $5 \times 1$ grid can be coloured so that exactly 3 cells are red and 2 cells are blue. These 10 colourings are shown to the right.

(b) We begin by observing that there are exactly 2 different ways to colour each of the 13 cells (red or blue), and so there are a total of $2^{13}$ possible colourings of a $1 \times 13$ grid.

For each $1 \times 13$ grid, Carrie counts the number of cells coloured red, call this number $r$, and she counts the number of cells coloured blue, call this number $b$.

Since there are 13 cells in total, then $r + b = 13$ and either $r > b$ or $b > r$ ($r$ and $b$ cannot be equal since they are integers and their sum is odd).

If $r > b$, then $r > 13 - r$ which gives $2r > 13$ or $r > 6.5$ and so $r \geq 7$ (since $r$ is an integer).

If $b > r$, then $b > 13 - b$ which gives $2b > 13$ or $b > 6.5$ and so $b \geq 7$ (since $b$ is an integer).

Thus, if the number of cells coloured red is greater than the number of cells coloured blue, then Carrie writes down the number of red cells (since she writes the maximum of $r$ and $b$), and this number is at least 7.

Alternatively, if the number of cells coloured blue is greater than the number of cells coloured red, then Carrie writes down the number of blue cells, and this number is at least 7.

In either case, each number that Carrie writes down in her list is between 7 and 13, inclusive.

At least one of the possible $2^{13}$ different colourings of a $1 \times 13$ grid has 7 cells coloured red and 6 cells coloured blue, and so Carrie’s list includes at least one 7.

Since Carrie’s list includes a 7 and every number in her list is 7 or greater, then the smallest number in her list is 7.

(c) In a $3 \times n$ grid, each column contains exactly 3 cells.

Each of these 3 cells can be coloured in one of two ways, either red or blue.

Since there are two ways that each of the 3 cells may be coloured, each column in a $3 \times n$ grid can be coloured in one of $2^3 = 8$ different ways.

Thus, it is possible to colour a $3 \times n$ grid for integers $n$, $1 \leq n \leq 8$, so that each column is coloured in a different way (such an example of a $3 \times 8$ grid is shown).

Since there are only 8 different ways to colour the 3 cells in any column, then every colouring of a $3 \times 9$ grid must have at least two columns that are coloured in an identical way. Thus, the smallest possible value of $n$ is 9.

(d) The given statement is true.

In a $5 \times 41$ grid, there are 5 rows and so each of the 41 columns has 5 cells.

In each column, at least 3 of the 5 cells must be the same colour.

In each column, either there are more cells coloured red than blue or there are more cells coloured blue than red (they can’t be equal in number since 5 is odd).
In each column, if there are more cells coloured red than blue, we call that column a red column.
Alternatively, if there are more cells coloured blue than red, we call that column a blue column.
Let the total number of red columns be \( R \) and the number of blue columns be \( B \), and so \( R + B = 41 \).
Similar to the argument in part (b), if \( R > B \), then \( R \geq 21 \), otherwise \( B \geq 21 \).
Assume that \( R \geq 21 \) (the argument that follows can be made in a similar way if \( B \geq 21 \)).
In this case, each of these 21 (or more) columns has more cells coloured red than those that are coloured blue and so each of these columns has at least 3 cells coloured red.
We will show that the location of 3 cells coloured red is the same in at least 3 of 21 columns.
Of all red columns, consider the first 21 (there are at least 21).
Moving from top to bottom within each of these red columns, consider the first 3 red cells (there are at least 3 cells coloured red).
What is the maximum number of ways to colour each column so that exactly 3 cells are coloured red?
Since a \( 5 \times 1 \) grid is the size of each column in a \( 5 \times 41 \) grid, this is the question that was answered in part (a), and so there are 10 such ways.
In the 21 red columns, assume that at most 2 columns have the same 3 cells coloured red.
Since there are only 10 different ways to colour each column so that exactly 3 cells are coloured red, then there are at most \( 2 \times 10 = 20 \) such columns.
However, there are 21 red columns and so we have a contradiction.
Thus, our assumption that in the 21 red columns, there are at most 2 columns in which the same 3 cells are coloured red was incorrect, and so there must be at least 3 columns which have the same 3 cells that are coloured red.
The 9 cells located at the intersection of these 3 columns and the 3 rows containing the red cells within these columns are all coloured red.